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Modelling Round-off Error in the Fast Gradient Method for Predictive Control

Ian McInerney[1], Eric C. Kerrigan[1,2], George A. Constantinides[1]

[1]Department of Electrical and Electronic Engineering, Imperial College London
[2]Department of Aeronautics, Imperial College London

Optimal Control Problem

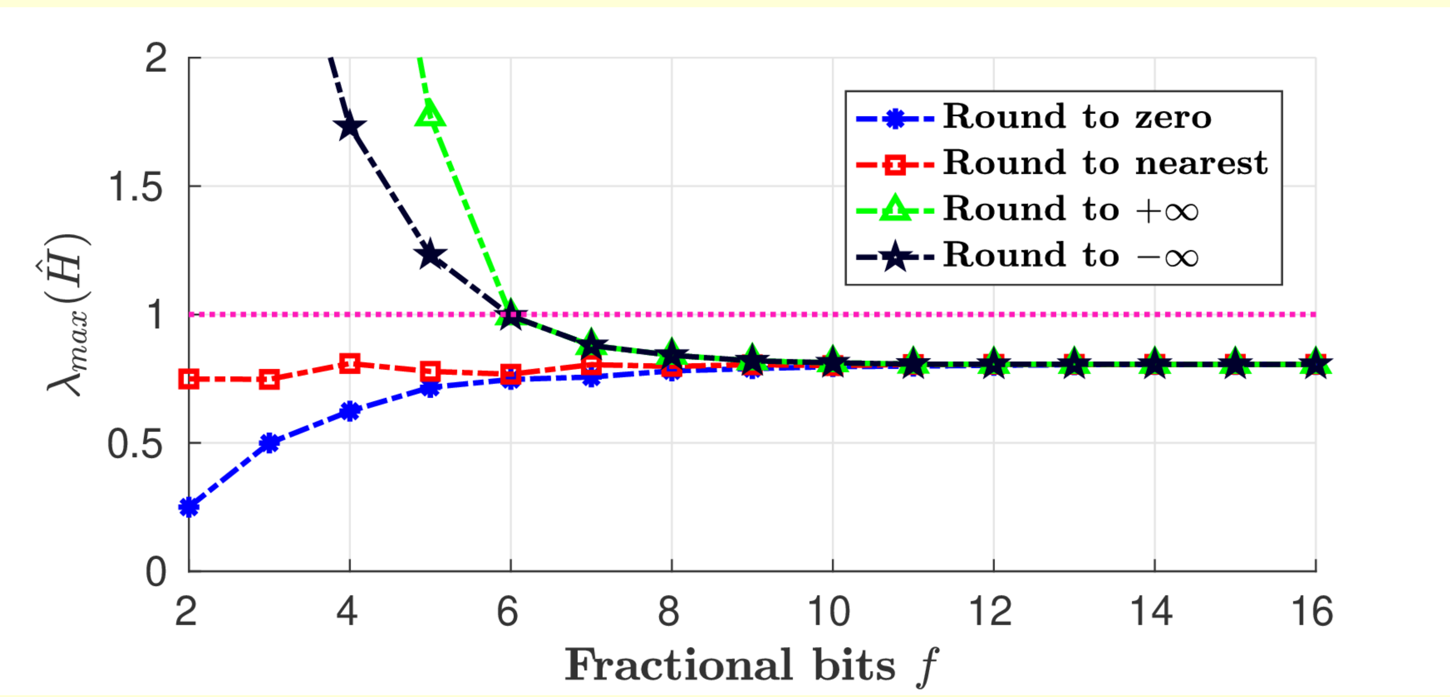
- Constrained Linear Quadratic Regulator

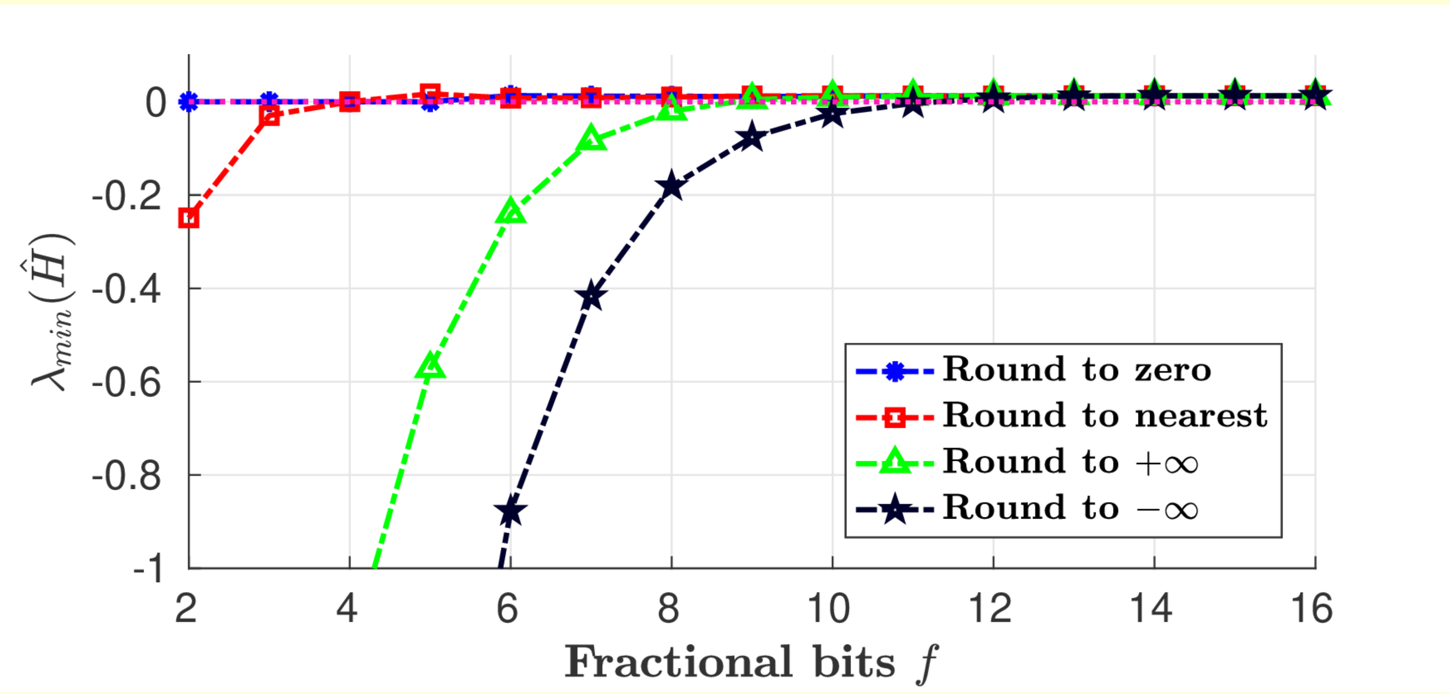
$$\min_{u,x} \frac{1}{2} x_N' P x_N + \frac{1}{2} \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$
$$\text{s.t. } x_{k+1} = Ax_k + Bu_k,\ k = 0, \dots N-1$$
$$Fu_k \leq c_u,\ k = 0, \dots N-1$$

Remove state variables

$$\min_u \frac{1}{2} u' H_c u + \hat{x}_0' J' u$$
$$\text{s.t. } Gu \leq F\hat{x}_0 + g$$

- Solve using the Fast Gradient Method
  - For stability and convexity: $\lambda(H_c) \in (0,1)$
- Converting problem to fixed-point representation can cause loss of convexity and stability





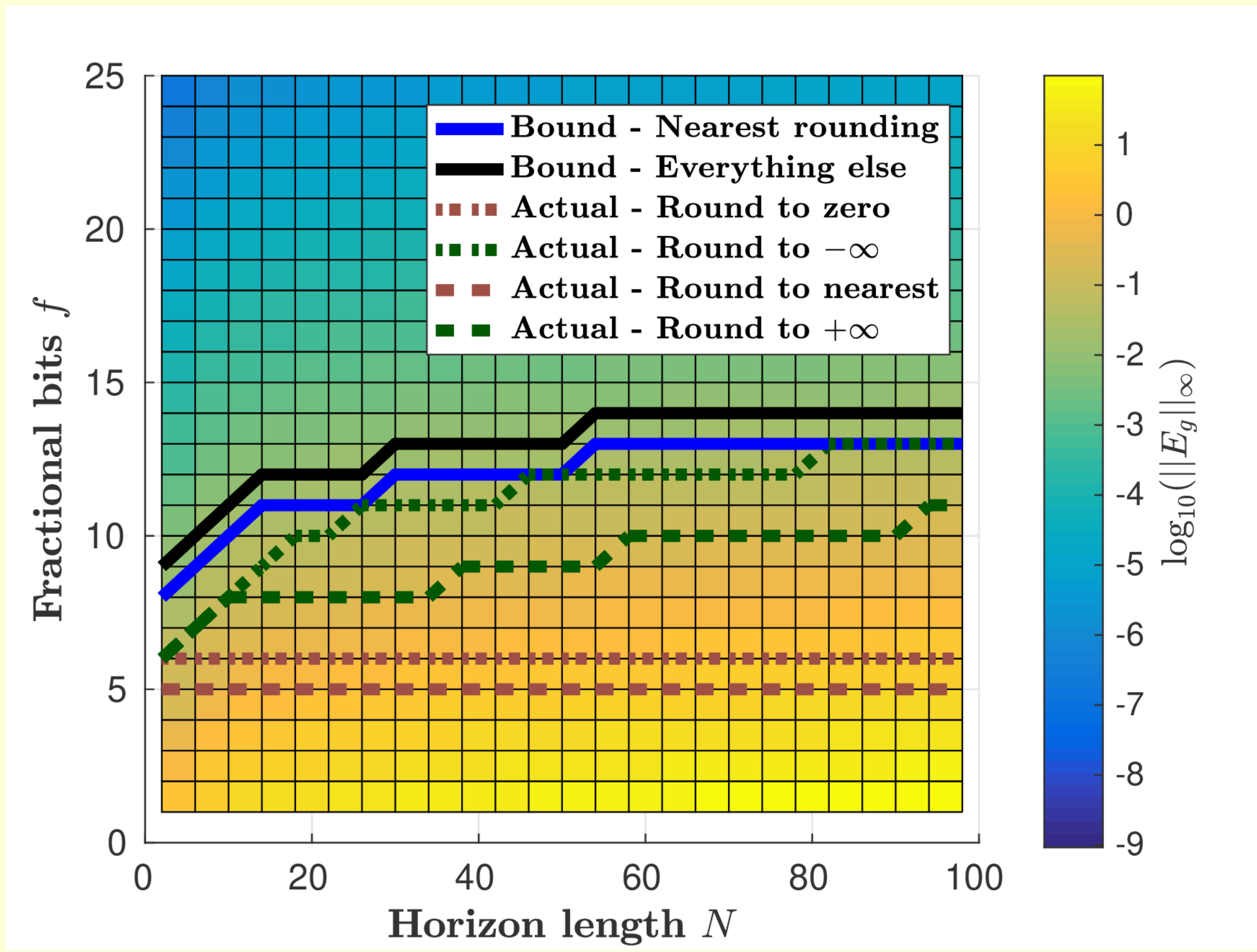
Rounding Model

- Model rounding loss as a perturbation matrix, and define the largest allowable perturbation

Definition 1 (Rounding stability margin)**.**
Let $\hat{H} = H + E$ with $\|E\|_2 = \beta$ and $\lambda(H) \in (0,1)$. The rounding stability margin η is the smallest value of β that causes the eigenvalues of $\hat{H}$ to leave the interval $(0,1)$.

- Use the pseudospectrum of H_c to compute η
  - For generic matrices, add $\pm\varepsilon$ to all terms as the perturbation, giving
    $$\eta(H) = \min\left\{ \|(-H)^{-1}\|_2^{-1}, \|(I-H)^{-1}\|_2^{-1} \right\}.$$
  - For Schur-stable systems, exploit the Toeplitz structure of H_c, and use its matrix symbol, giving
    $$\eta(H) = \min\left\{ \|(-\mathcal{P}_H)^{-1}\|_{H_\infty}^{-1}, \|(I_m - \mathcal{P}_H)^{-1}\|_{H_\infty}^{-1} \right\},$$
    $$\mathcal{P}_{H_{cP}}(z) := (\, z\mathcal{G}(z)_s)^* Q(\, z\mathcal{G}(z)_s) + R \quad \forall z \in \{z \in \mathbb{C} : |z| = 1\}.$$

- For a system with m inputs and a horizon of length N, size the fractional bits as follows:
  - For generic matrices:
    $$f = \begin{cases} \left\lceil -\log_2\left(\frac{\eta}{mN}\right)\right\rceil - 1 & \text{if using round to nearest,} \\ \left\lceil -\log_2\left(\frac{\eta}{mN}\right)\right\rceil & \text{otherwise.} \end{cases}$$
  - For Schur-stable predicted systems using round to nearest or round to zero, solve:
    $$\min \quad f$$
    $$\text{s.t.} \quad |\varepsilon_f| m(2k-1) + 2\|\mathcal{P}_{\bar{H}}(k,\cdot)\|_{H_\infty} < \eta$$
    where
    $$\mathcal{G}_P := \begin{cases} x^+ = Ax + Bu \\ y = B'Px \end{cases},\ \mathcal{P}_n(z) := \sum_{i=0}^{n-1} A^i z^{-i} \quad \forall z \in \{z \in \mathbb{C} : |z| = 1\}$$
    $$\mathcal{P}_{\bar{H}}(n,z) := z\mathcal{G}_P(z) - B'P\mathcal{P}_n(z)B \quad \forall z \in \{z \in \mathbb{C} : |z| = 1\}.$$



Minimum fractional bits needed for a given horizon length

- Exploiting the Toeplitz structure leads to a reduction of the number of fractional bits by 40% compared to generic matrices.
- The minimum fractional length uses 77% less memory, 33% fewer DSP blocks, and is 25% faster compared with the floating-point version

Fractional Length	Logic Resources				Power (mW)	Solve Time (µS)
	LUT	FF	DSP	BRAM		
f=12	947	768	4	2	20	532.17
f=16	1,136	912	4	2	25	612.17
f=21	887	1,033	8	8	43	701.77
f=26	993	1,237	12	9	48	701.77
Float (single)	2,161	1,545	5	14	51	982.17

Resource usage for the Fast Gradient Method on a Zynq 7020