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HiPEDS

Bounding Computational Complexity under Cost Function Scaling in Predictive Control

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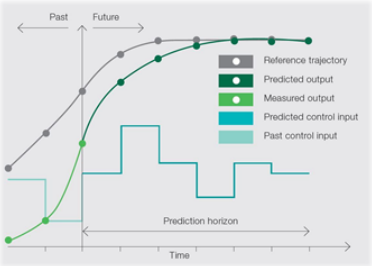
Outline

Model Predictive Control



$$\min_{u,x} \frac{1}{2} x_N' P x_N + \frac{1}{2} \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k, \;\; k = 0, \ldots, N-1$$

$$x_0 = \bar{x}_0$$

$$Eu_k \leq c_u, \;\; k = 0, \ldots, N-1$$

$$\min_{u} \frac{1}{2} u' H u + \bar{x}_0' J' u$$

$$\text{s.t. } Gu \leq g$$

Fast Gradient Method

- Accelerated first-order optimization solver
- Project onto the constraint set
- Constant step-size
- Maximum iterations as termination criteria

Algorithm 1 Fast gradient method for the solution of MPC problem (6) at state x (*optimized for parallel hardware*)

Require: Initial iterate $z_0 \in \mathbb{K}$, $y_0 = z_0$, upper (lower) bound L ($\mu > 0$) on maximum (minimum) eigenvalue of Hessian H_F, step size $\beta = (\sqrt{L} - \sqrt{\mu})/(\sqrt{L} + \sqrt{\mu})$

for $i = 0$ to $I_{\max} - 1$ **do**

  $t_i := (I - (1/L)H_F)y_i - (1/L)\Phi x$

  $z_{i+1} := \pi_{\mathbb{K}}(t_i)$

  $y_{i+1} := (1 + \beta)z_{i+1} - \beta z_i$

end for

Matrix Structures

The condensed MPC problem has Hessian

$$H := \Gamma' \bar{Q} \Gamma + \bar{S}' \Gamma + \Gamma' \bar{S} + \bar{R}$$

with

$$\Gamma = \begin{bmatrix} B & 0 & 0 & & 0 \\ AB & B & 0 & & 0 \\ A^2 B & AB & B & & 0 \\ \vdots & & & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \cdots & B \end{bmatrix}$$

Dense Prediction Matrix

$$\Gamma = \begin{bmatrix} B & 0 & 0 & & 0 \\ AB & B & 0 & & 0 \\ A^2B & AB & B & & 0 \\ \vdots & & & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \cdots & B \end{bmatrix}$$

When $\mathcal{G}_s$ is Schur-stable, this is Toeplitz with the matrix symbol

$$\mathcal{P}_\Gamma(z) = \sum_{i=0}^{\infty} A^i B z^{-i} = z(zI - A)^{-1}B = z\mathcal{G}_s(z) \qquad \forall z \in \mathbb{T}$$

Condensed Hessian Matrix

Topelitx Structure (No cross terms)

The Hessian matrix without S terms (defined as H_{cP}) is a Toeplitz matrix with the matrix symbol

$$\mathcal{P}_{H_{cP}}(z) := \mathcal{P}_{\Gamma}(z)^* Q \mathcal{P}_{\Gamma}(z) + R$$

Spectral Bound (No cross terms)

Bound the eigenvalues of the matrix (and the condition number) using the eigenvalues of the symbol

$$\lambda_{min}(\mathcal{P}_{H_{cP}}) \leq \lambda(H_{cP}) \leq \lambda_{max}(\mathcal{P}_{H_{cP}})$$

Condensed Hessian Matrix

Matrix Structure (With cross terms)

The Hessian matrix with S terms (H_{cS}) can be written as

$$H_{cS} = H_n - H_e$$

where H_n is a Toeplitz matrix

$$H_n := H_{cP} + (I_N \otimes S)'\Gamma + \Gamma'(I_N \otimes S)$$

and H_e is a correction term

$$H_e := S_c'\Gamma + \Gamma' S_c$$

$$S_c := \begin{bmatrix} I_{N-1} \otimes 0 & 0 \\ 0 & S \end{bmatrix}$$

Condensed Hessian Matrix

Spectrum of H_n

H_n is Toeplitz with symbol

$$\mathcal{P}_{H_n}(z) = \mathcal{P}_{H_{cQ}}(z) + S'\mathcal{P}_{\Gamma}(z) + \mathcal{P}_{\Gamma}^{*}(z)S \qquad \forall z \in \mathbb{T}$$

Spectrum of H_e

The non-zero eigenvalues of H_e are the same as the eigenvalues of

$$U := \begin{bmatrix} B'S & I \\ S'W_cS & S'B \end{bmatrix}$$

where W_c is the controllability Gramian of $\mathcal{G}_s$.

Condensed Hessian Matrix

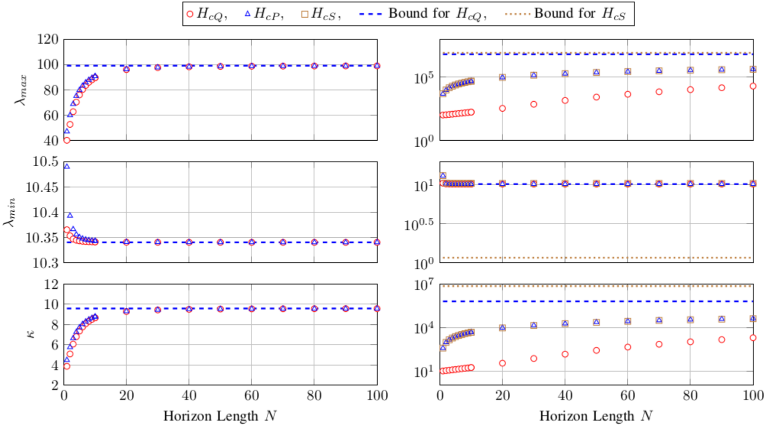
Spectral Bound (With cross terms)

Bound the eigenvalues of the matrix using eigenvalue inequalities

$$\gamma := \lambda_{max}(\mathcal{P}_{H_n}), \qquad \beta := \lambda_{min}(\mathcal{P}_{H_n})$$
$$\eta := \lambda_{max}(U), \qquad \nu := \lambda_{min}(U)$$

$$\max\{0, \beta - \eta\} \leq \lambda(H_{cS}) \leq \gamma - \nu$$

Condensed Hessian Matrix



(a) System 1

(b) System 2

Cost Function Scaling

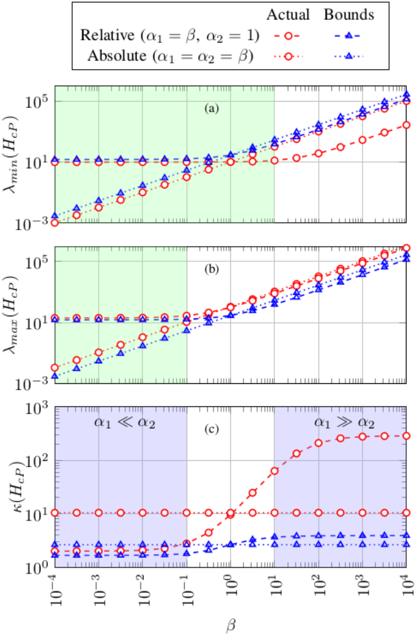
Examine how the problem's computational complexity changes when the cost function is scaled using $\hat{Q} := \alpha_1 Q$, $\hat{R} := \alpha_2 R$.

Condition Number Bound

$$\kappa(\hat{H}_c) \geq 1 + 2\frac{\sqrt{\alpha_1^2 n_1 + 2\alpha_1\alpha_2 n_2 + \alpha_2^2 n_3}}{\alpha_1 \|\mathcal{G}_Q\|_{H_2}^2 + \alpha_2 \|R^{1/2}\|_F^2},$$

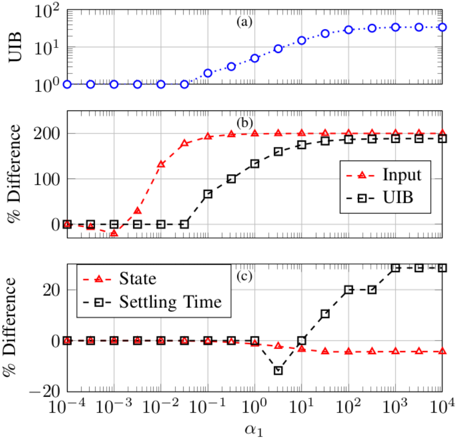
with

$$\begin{aligned} n_1 &:= m I_6 - \|\mathcal{G}_Q\|_{H_2}^4, \\ n_2 &:= m\|\mathcal{G}_{QR}\|_{H_2}^2 - \|\mathcal{G}_Q\|_{H_2}^2 \|R^{1/2}\|_F^2, \\ n_3 &:= m\|R\|_F^2 - \|R^{1/2}\|_F^4 \end{aligned}$$



Actual Bounds
Relative ($\alpha_1 = \beta$, $\alpha_2 = 1$)
Absolute ($\alpha_1 = \alpha_2 = \beta$)
(a)
$\lambda_{min}(H_{cP})$
(b)
$\lambda_{max}(H_{cP})$
(c)
$\alpha_1 \ll \alpha_2$
$\alpha_1 \gg \alpha_2$
$\kappa(H_{cP})$
β

FGM Scaling


UIB
% Difference
% Difference
(a)
(b)
(c)
Input
UIB
State
Settling Time
10^2
10^1
10^0
200
100
0
20
0
-20
10^{-4} 10^{-3} 10^{-2} 10^{-1} 10^0 10^1 10^2 10^3 10^4
α_1

Preconditioning

Extending Prior Analysis

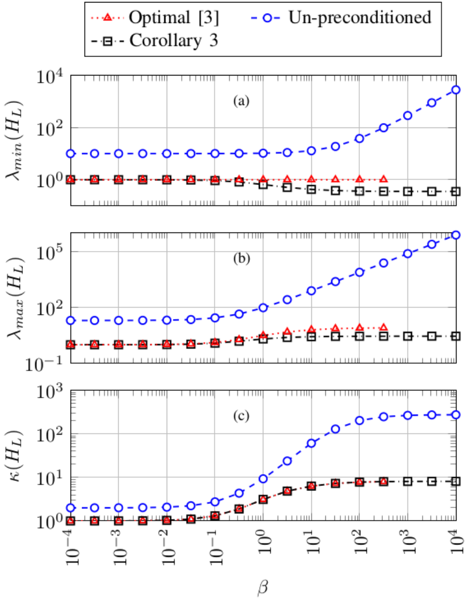
The prior results are also valid when symmetric (left-right) preconditioners are used to form

$$H_{pre} = L^{-1} H_c (L^{-1})'$$

Preconditioner Design

Toeplitz methods can be used to design the preconditioner L, where L is the lower-triangular Cholesky-decomposition of

$$M := B'PB + S'B + B'S + R.$$



Optimal [3]
Un-preconditioned
Corollary 3
(a)
$\lambda_{min}(H_L)$
(b)
$\lambda_{max}(H_L)$
(c)
$\kappa(H_L)$
β

Summary

- Derived a relation between the spectrum of MPC matrices and the system transfer function
- Examined the change in computational complexity as problem varied
- Derived a novel closed-form preconditioner

Future Work

- Horizon-dependent spectral bounds by relaxing the assumption of Schur-stability

Paper available at arXiv:1902.02221v1 [math.OC]