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Modeling Round-off Error in the Fast Gradient Method for Predictive Control

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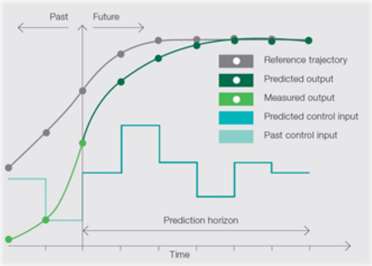
Outline

1. Preliminaries
2. Generic round-off error model
3. Parametric round-off error model
4. Conclusions

Model Predictive Control



$$\min_{u,x} \frac{1}{2} x_N' P x_N + \frac{1}{2} \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k$$

$$\text{s.t. } x_{k+1} = A x_k + B u_k, \ \ k = 0, \dots, N-1$$

$$x_0 = \bar{x}_0$$

$$E u_k \leq c_u, \ \ k = 0, \dots, N-1$$

$$\min_{u} \frac{1}{2} u' H u + \bar{x}_0' J' u$$

$$\text{s.t. } Gu \leq g$$

Fast Gradient Method

- Accelerated first-order optimization solver
- Project onto the constraint set
- Necessary condition for stability: $\lambda(H) \in (0, 1)$

for $i = 0$ to $I_{\max} - 1$ **do**
  $t_i := (I - \frac{1}{L}H)y_i - \frac{1}{L}\Phi x$
  $z_{i+1} := \pi_{\mathbb{K}}(t_i)$
  $y_{i+1} := (1 + \beta)z_{i+1} - \beta z_i$
end for

Fixed-Point Arithmetic

- Quantize $\mathbb{R}$ into $\mathbb{Z}$ by multiplying by a scale factor and rounding

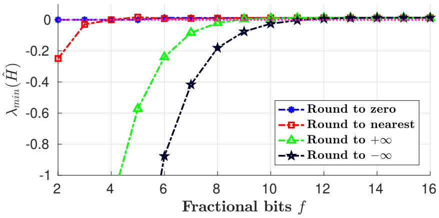
$$\overbrace{18020.017052}^{\mathbb{R}} \xrightarrow{\times 10^5} \overbrace{\underbrace{18020}_{\text{integer}}\,\underbrace{01705}_{\text{fraction}}}^{\text{Fixed-point}}$$

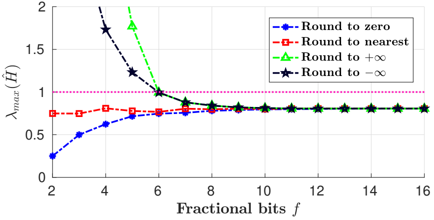
- Only requires integer computations in hardware
- Introduces ϵ_f round-off error

Model rounding action as an additive disturbance matrix:

$$\hat{H} = H + E$$

Rounding Stability Margin





Rounding Stability Margin (η)

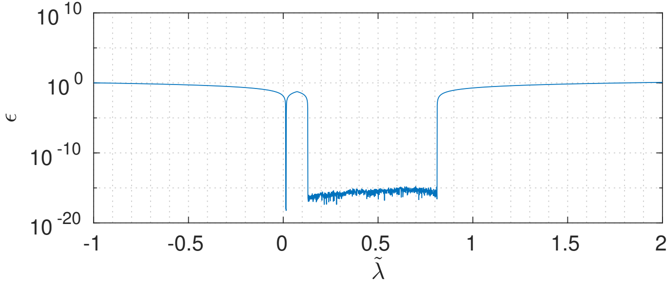
The smallest value of $\|E\|_2$ that causes the eigenvalues of $\hat{H}$ to leave the interval $(0, 1)$.

Rounding Stability Margin

Computation

Use the ϵ-pseudospectrum of H at $\tilde{\lambda} = \{0, 1\}$.

$$\left\|(\tilde{\lambda} I - H)^{-1}\right\| \geq \frac{1}{\epsilon} \xrightarrow{\text{Equivalently}} \tilde{\lambda} \in \lambda(H + E) \text{ with } \|E\| \leq \epsilon$$



Generic Rounding Model

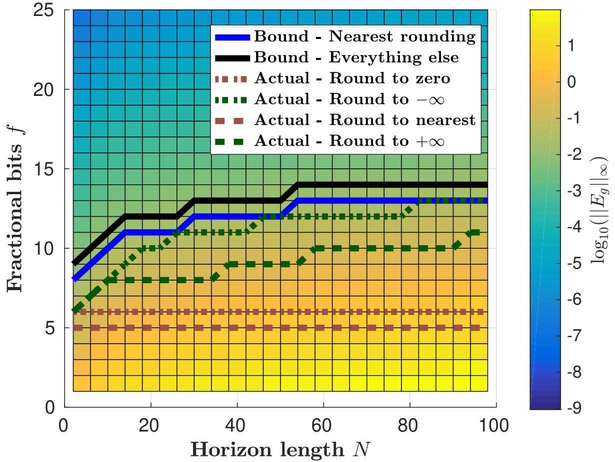
Model

$$E_g := \begin{bmatrix} \pm\epsilon_f & \pm\epsilon_f & \dots \\ \pm\epsilon_f & \pm\epsilon_f & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Fractional Bits Necessary for Stability

$$f = \begin{cases} \left\lceil -\log_2\left(\frac{\eta}{mN}\right)\right\rceil - 1 & \text{if using round to nearest,} \\ \left\lceil -\log_2\left(\frac{\eta}{mN}\right)\right\rceil & \text{otherwise.} \end{cases}$$

Results


Bound - Nearest rounding
Bound - Everything else
Actual - Round to zero
Actual - Round to $-\infty$
Actual - Round to nearest
Actual - Round to $+\infty$
Fractional bits f
Horizon length N
$\log_{10}(||E_g||_\infty)$
0
5
10
15
20
25
0
20
40
60
80
100
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9

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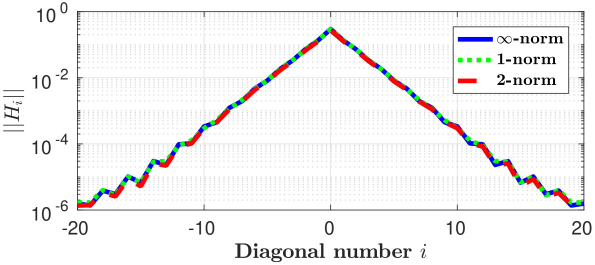
Hessian Structure

Toeplitz Components

$$H_i = \begin{cases} B'(A^i)'PB & \text{if } i > 0, \\ B'PB + R & \text{if } i = 0, \\ B'PA^{|i|}B & \text{if } i < 0. \end{cases}$$

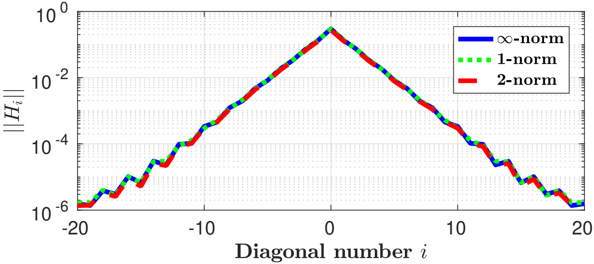


Parametric Rounding Model

Idea

Split round-off error model into two parts: $E_p := E_G + E_T$

$$(E_G)_i := \begin{cases} E_g & \text{if } i < k, \\ 0 & \text{otherwise,} \end{cases} \qquad (E_T)_i := \begin{cases} H_i & \text{if } |i| \geq k, \\ 0 & \text{otherwise.} \end{cases}$$



Parametric Rounding Model

Fractional Width Selection

Choose ϵ_f such that

$$|\epsilon_f| m(2k-1) + 2\|\mathcal{P}_{\bar{H}}(k,\cdot)\|_{H_\infty} < \eta,$$

where

$$\mathcal{P}_{\bar{H}}(n,z) := z\mathcal{G}_P(z) - B'P\mathcal{P}_n(z)B \quad \forall z \in \mathbb{T},$$

$$\mathcal{G}_P := \begin{cases} x^+ = Ax + Bu \\ y = B'Px \end{cases},$$

$$\mathcal{P}_n(z) := \sum_{i=0}^{n-1} A^i z^{-i} \quad \forall z \in \mathbb{T}.$$

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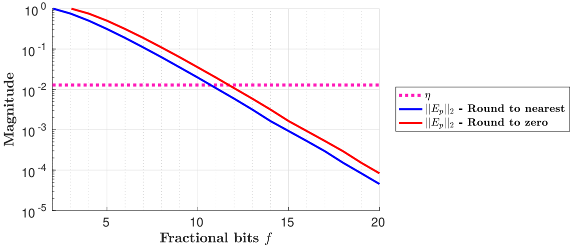
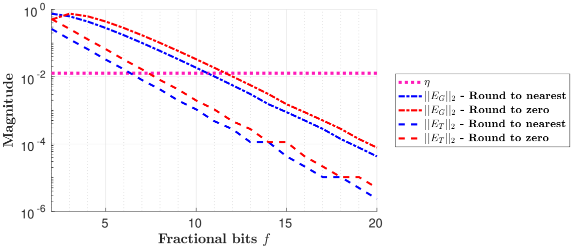
Results





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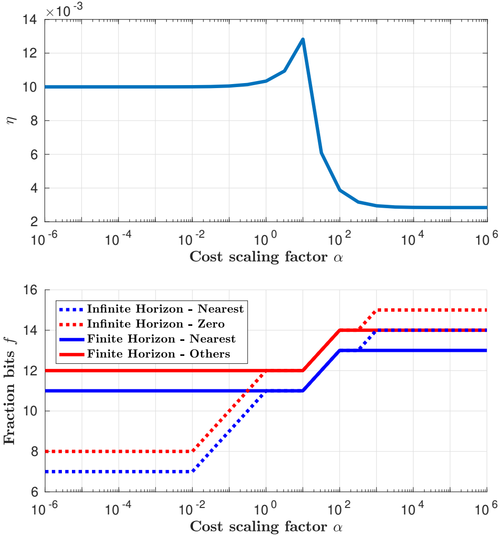
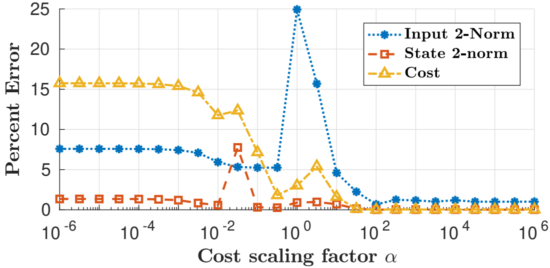
Results



Results



Fractional Length	Logic Resources LUT	FF	DSP	BRAM	Power (mW)	Solve Time (μs)
f=12	947	768	4	2	20	532.17
f=16	1,136	912	4	2	25	612.17
f=21	887	1,033	8	8	43	701.77
f=26	993	1,237	12	9	48	701.77

Conclusions

Key Contributions

- Compute the Rounding Stability Margin for the Fast Gradient Method using the ϵ-pseudospectrum
- Exploit Toeplitz structure to model exact round-off error
- Reduce the fractional bits needed by 30–45%
- Reduction in hardware usage and solution time by up to 77% and 25% respectively

Future Directions

- Derive sufficient condition for stability of the Fast Gradient Method
- Bound perturbation of optimal vector u^* when problem is quantized