Closed-Form Preconditioner Design for Linear Predictive Control

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Presented at the 21st IFAC World Congress

Preconditioner Features

- Horizon-independent
- Comparable performance to existing SDP-based methods
- Required design computations scale with number of states and inputs, but not horizon length
- Exploits the Toeplitz structure of the condensed Hessian for problems with Schur-stability

Outline

Model Predictive Control

$$\min_{u,x} \frac{1}{2} x_N' P x_N + \frac{1}{2} \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}' \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k, \;\; k = 0, \ldots, N-1$$

$$x_0 = \bar{x}_0$$

$$Eu_k \leq c_u, \;\; k = 0, \ldots, N-1$$

$$H_{cP} := \Gamma' \bar{Q} \Gamma + \bar{R}$$

with

$$\Gamma := \begin{bmatrix} B & 0 & 0 & & 0 \\ AB & B & 0 & & 0 \\ A^2B & AB & B & & 0 \\ \vdots & & & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \cdots & B \end{bmatrix},$$

$$\bar{Q} := \begin{bmatrix} I_{N-1} \otimes Q & 0 \\ 0 & P \end{bmatrix}, \quad \bar{R} := I_N \otimes R$$

$$\min_{u} \frac{1}{2} u' H_{cP} u + \bar{x}_0' J' u$$

$$\text{s.t. } u \in \mathbb{K} := \{u : Gu \leq g\}$$

Preconditioning First-order Methods

- Algorithm convergence rate dependent on problem conditioning
- Problem conditioning equivalent to the condition number of the Hessian
- Commonly applied as a symmetric preconditioner $L_N^{-1} H_{cP} (L_N^{-1})'$

Preconditioning Strategies

- Matrix equilibration
- Solve a SDP to find the L_N that minimizes the resulting condition number

Dense Prediction Matrix

$$\Gamma = \begin{bmatrix} B & 0 & 0 & & 0 \\ AB & B & 0 & & 0 \\ A^2B & AB & B & & 0 \\ \vdots & & & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \cdots & B \end{bmatrix}$$

When $\mathcal{G}_s$ is Schur-stable, this is Toeplitz with the matrix symbol

$$\mathcal{P}_\Gamma(z) = \sum_{i=0}^{\infty} A^i B z^{-i} = z(zI - A)^{-1}B = z\mathcal{G}_s(z) \qquad \forall z \in \mathbb{T}$$

Condensed Hessian Matrix

Topelitx Structure

The Hessian matrix H_{cP} is a Toeplitz matrix with the matrix symbol

$$H_{cP} := \Gamma'\bar{Q}\Gamma + \bar{R} \quad \Leftrightarrow \quad \mathcal{P}_{H_{cP}}(z) := \mathcal{P}_{\Gamma}(z)^{*} Q \mathcal{P}_{\Gamma}(z) + R$$

Spectral Bound

Bound the eigenvalues of the matrix (and the condition number) using the eigenvalues of the symbol

$$\lambda_{min}(\mathcal{P}_{H_{cP}}) \leq \lambda(H_{cP}) \leq \lambda_{max}(\mathcal{P}_{H_{cP}})$$

$$\lim_{N \to \infty} \kappa(H_{cP}) = \kappa(\mathcal{P}_{H_{cP}})$$

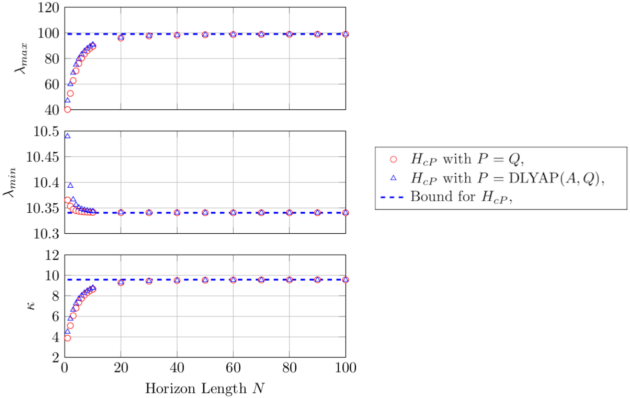
Condensed Hessian Matrix

Example: 4-state, 2-input system

$$x^+ = \begin{bmatrix} 0.7 & -0.1 & 0.0 & 0.0 \\ 0.2 & -0.5 & 0.1 & 0.0 \\ 0.0 & 0.1 & 0.1 & 0.0 \\ 0.5 & 0.0 & 0.5 & 0.5 \end{bmatrix} x + \begin{bmatrix} 0.0 & 0.1 \\ 0.1 & 1.0 \\ 0.1 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} u,$$

$Q = \text{diag}(10, 20, 30, 40),$

$R = \text{diag}(10, 20)$



Analysis of preconditioned matrix

- Use a block-diagonal symmetric preconditoner:

$$H_L = L_N^{-1} H_{cP} (L_N^{-1})'$$
$$L_N = I_N \otimes L$$

Results

- H_L is Toeplitz with $\mathcal{P}_{H_L} := L^{-1} \mathcal{P}_{H_{cP}} (L^{-1})'$
- Previous spectral bounds extend to preconditioned matrix for $\mathcal{P}_{H_L}$ instead of $\mathcal{P}_{H_{cP}}$

Closed-form preconditioner

Preconditioner Design

The matrix H_{cP} can be symmetrically preconditioned as $L_N^{-1} H_{cP} (L_N^{-1})'$, where the blocks L are the lower-triangular Cholesky decomposition of M with

$$M := B'PB + R,$$

and P is the solution to the Lyapunov equation

$$A'PA + Q = P.$$

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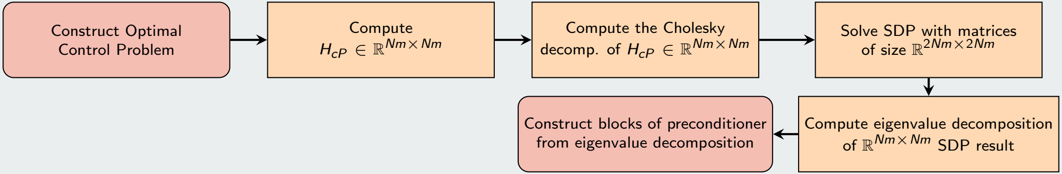
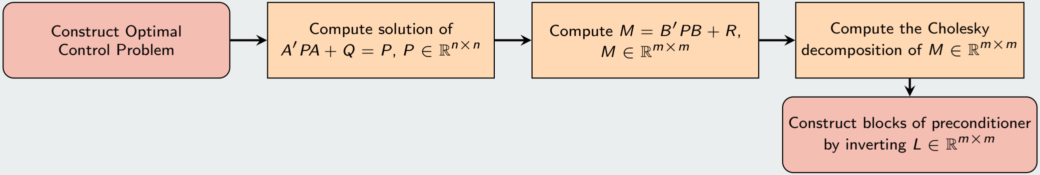
Preconditioner design flow

SDP-based design flow



Proposed closed-form design flow



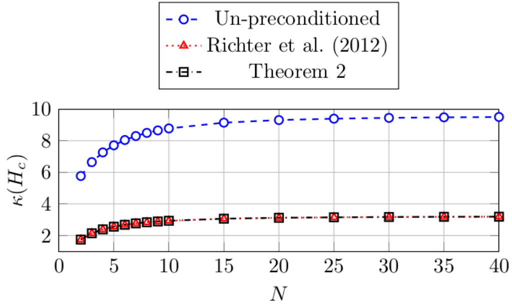
Closed-form preconditioner - results

Equivalent performance to SDP-based preconditioners

Example: 4-state, 2-input system

$$x^+ = \begin{bmatrix} 0.7 & -0.1 & 0.0 & 0.0 \\ 0.2 & -0.5 & 0.1 & 0.0 \\ 0.0 & 0.1 & 0.1 & 0.0 \\ 0.5 & 0.0 & 0.5 & 0.5 \end{bmatrix} x + \begin{bmatrix} 0.0 & 0.1 \\ 0.1 & 1.0 \\ 0.1 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} u,$$

$$Q = \mathrm{diag}(10, 20, 30, 40),$$
$$R = \mathrm{diag}(10, 20),$$
$$P = Q$$



Closed-form preconditioner - results

Examine H_{cP} at different β values of:

$$\min_{u,x} \frac{1}{2} x_N' P x_N + \frac{1}{2} \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}' \begin{bmatrix} \beta Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

$$\text{s.t. } x_{k+1} = A x_k + B u_k, \quad k = 0, \ldots, N-1$$

$$x_0 = \bar{x}_0$$

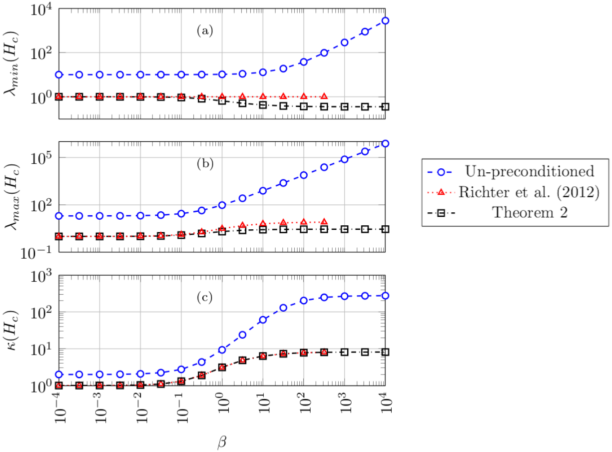
with

$$x_{k+1} = \begin{bmatrix} 0.7 & -0.1 & 0.0 & 0.0 \\ 0.2 & -0.5 & 0.1 & 0.0 \\ 0.0 & 0.1 & 0.1 & 0.0 \\ 0.5 & 0.0 & 0.5 & 0.5 \end{bmatrix} x_k + \begin{bmatrix} 0.0 & 0.1 \\ 0.1 & 1.0 \\ 0.1 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} u_k$$

$$Q = \text{diag}(10, 20, 30, 40),$$

$$R = \text{diag}(10, 20),$$

$$P = Q$$



Conclusions

Key Contributions

- Closed-form & horizon independent preconditioner
- Equivalent performance to SDP-based preconditioners

Future Directions

- Extend results to non-Schur-stable systems
- Apply preconditioner to first-order methods to measure performance