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Circulant Preconditioning of the Fast Gradient Method for Predictive Control

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Preconditioner Features

- Horizon-independent
- Comparable performance to existing SDP-based methods
- Required design computations scale with number of states and inputs, but not horizon length
- Exploits the Toeplitz structure of the condensed Hessian

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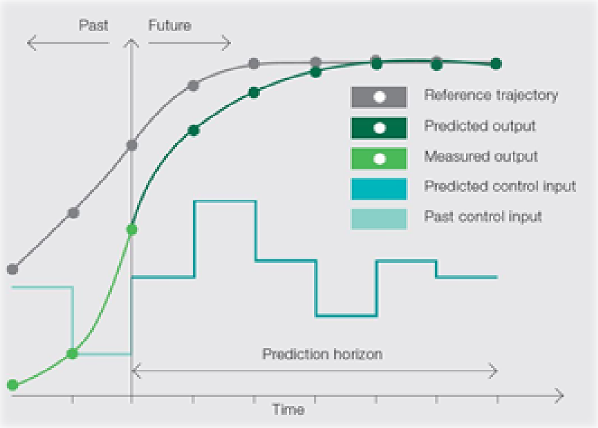
Model Predictive Control



Model Predictive Control

$$\min_{u,x} \frac{1}{2} x_N' P x_N + \frac{1}{2} \sum_{k=0}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}' \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k, \; k = 0, \ldots, N-1$$

$$x_0 = \bar{x}_0$$

$$Eu_k \leq c_u, \; k = 0, \ldots, N-1$$

$$H := \Gamma' \bar{Q} \Gamma + \bar{R}$$

with

$$\Gamma := \begin{bmatrix} B & 0 & 0 & & 0 \\ AB & B & 0 & & 0 \\ A^2B & AB & B & & 0 \\ \vdots & & & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \cdots & B \end{bmatrix},$$

$$\bar{Q} := \begin{bmatrix} I_{N-1} \otimes Q & 0 \\ 0 & P \end{bmatrix}, \quad \bar{R} := I_N \otimes R$$

$$\min_u \frac{1}{2} u' H u + \bar{x}_0' J' u$$

$$\text{s.t. } u \in \mathbb{K} := \{u : Gu \leq g\}$$

Preconditioning First-order Methods

- Algorithm convergence rate dependent on problem conditioning
- Problem conditioning equivalent to the condition number of the Hessian
- Commonly applied as a symmetric preconditioner $L_N^{-1} H (L_N^{-1})'$

Preconditioning Strategies

- Matrix equilibration
- Solve a SDP to find the L_N that minimizes the resulting condition number

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Dense Prediction Matrix

$$\Gamma = \begin{bmatrix} B & 0 & 0 & & 0 \\ AB & B & 0 & & 0 \\ A^2B & AB & B & & 0 \\ \vdots & & & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \cdots & B \end{bmatrix}$$

When $\mathcal{G}_s$ is Schur-stable, this is Toeplitz with the matrix symbol

$$\mathcal{P}_\Gamma(z) = \sum_{i=0}^{\infty} A^i B z^{-i} = z(zI - A)^{-1}B = z\mathcal{G}_s(z) \qquad \forall z \in \mathbb{T}$$

What about non-Schur stable systems?

Introduce static-gain feedback controller

Let $u_k = Kx_k + v_k$, giving $A_c := A - BK$

$$\begin{aligned}\min_{v,x}\ & \frac{1}{2}x_N' P x_N + \frac{1}{2}\sum_{k=0}^{N-1}\begin{bmatrix}x_k\\ v_k\end{bmatrix}'\begin{bmatrix}Q_c & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}x_k\\ v_k\end{bmatrix}\\ \text{s.t. }\ & x_{k+1} = A_c x_k + Bv_k,\ \ k = 0,\dots,N-1\\ & x_0 = \bar{x}_0\\ & Fv_k \le c_v,\ \ k = 0,\dots,N-1\end{aligned}$$

$$H_c := \Gamma_c' \bar{Q}_c \Gamma_c + \Gamma_c' \bar{K}' \bar{R} + \bar{R}\bar{K}\Gamma_c + \bar{R}$$

with

$$\Gamma_c := \begin{bmatrix} B & 0 & 0 & & 0\\ A_c B & B & 0 & & 0\\ A_c^2 B & A_c B & B & & 0\\ \vdots & & & \ddots & \vdots\\ A_c^{N-1}B & A_c^{N-2}B & A_c^{N-3}B & \cdots & B\end{bmatrix},$$

$$Q_c := Q + K'RK', \quad \bar{K} := I_N \otimes -K$$

$$\bar{Q} := \begin{bmatrix} I_{N-1}\otimes Q_c & 0\\ 0 & P\end{bmatrix}, \quad \bar{R} := I_N \otimes R$$

Matrix symbols

Prestabilized prediction matrix

Is Toeplitz with the matrix symbol

$$\mathcal{P}_{\Gamma_c}(z) = \sum_{i=0}^{\infty} A_c^i B z^{-i} = z(zI - A_c)^{-1} B = z\mathcal{G}_c(z) \qquad \forall z \in \mathbb{T}$$

Prestabilized condensed Hessian

Is Toeplitz with the matrix symbol

$$H_c = \Gamma_c' \bar{Q}_c \Gamma_c + \Gamma_c' \bar{K}' \bar{R} + \bar{R}\bar{K}\Gamma_c + \bar{R}$$

$$\Downarrow$$

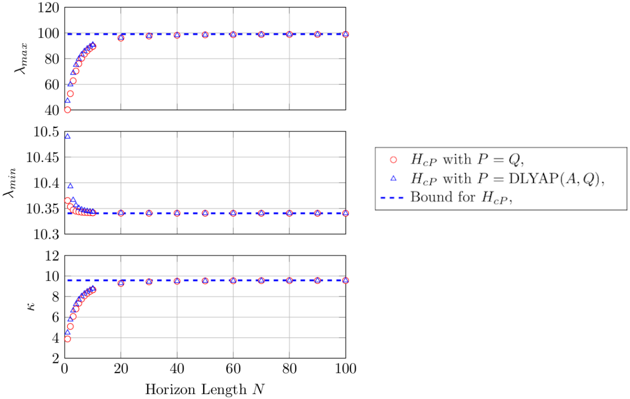
$$\mathcal{P}_{H_c}(z) := \mathcal{P}_{\Gamma_c}(z)^* Q_c \mathcal{P}_{\Gamma_c}(z) + \mathcal{P}_{\Gamma_c}(z)^* K' R + RK\mathcal{P}_{\Gamma_c}(z) + R \qquad \forall z \in \mathbb{T}$$

Spectral Bounds

Bound the eigenvalues of the Hessian (and the condition number) using the eigenvalues of the symbol

$$\lambda_{min}(\mathcal{P}_{H_c}) \leq \lambda(H_c) \leq \lambda_{max}(\mathcal{P}_{H_c})$$

$$\lim_{N\to\infty} \kappa(H_c) = \kappa(\mathcal{P}_{H_c})$$



Closed-form preconditioner

Goals

- Preserve separation of the feasible sets across the horizon
- Preconditioned Hessian is symmetric
- Improve performance

Closed-form preconditioner - Design

Strang's Preconditioner

When V is Toeptliz, $W^{-1}V$ clusters eigenvalues near 1 when W is the circulant completion of V.

Example:

$$V := \begin{bmatrix} V_0 & V_1 & V_2 & V_3 & V_4 \\ V_{-1} & V_0 & V_1 & V_2 & V_3 \\ V_{-2} & V_{-1} & V_0 & V_1 & V_2 \\ V_{-3} & V_{-2} & V_{-1} & V_0 & V_1 \\ V_{-4} & V_{-3} & V_{-2} & V_{-1} & V_0 \end{bmatrix}, \qquad W := \begin{bmatrix} V_0 & V_1 & V_2 & V_{-2} & V_{-1} \\ V_{-1} & V_0 & V_1 & V_2 & V_{-2} \\ V_{-2} & V_{-1} & V_0 & V_1 & V_2 \\ V_2 & V_{-2} & V_{-1} & V_0 & V_1 \\ V_1 & V_2 & V_{-2} & V_{-1} & V_0 \end{bmatrix}.$$

Closed-form preconditioner - Design

- Use a block-diagonal symmetric preconditoner:

$$H_L = L_N^{-1} H_c (L_N^{-1})'$$
$$L_N = I_N \otimes L$$

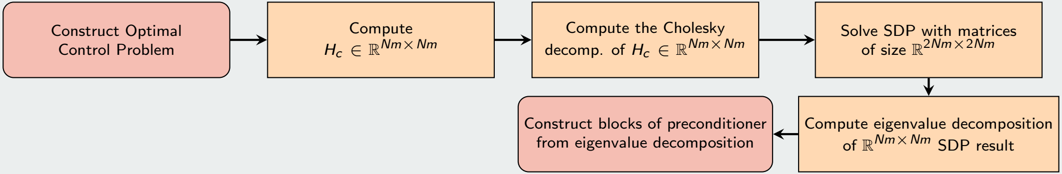
- Define L as Cholesky factor of the diagonal element of H_c

$$L := \mathrm{chol}(B'PB - B'K'R - RKB + R)$$

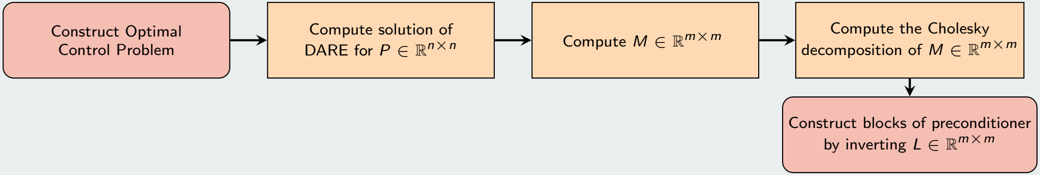
with P the solution to the Discrete-time Riccatti equation

Preconditioner design flow

SDP-based design flow

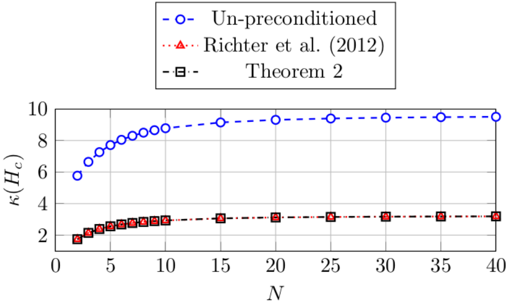


Proposed closed-form design flow



Closed-form preconditioner - results

- H_L is Toeplitz with $\mathcal{P}_{H_L} := L^{-1}\mathcal{P}_{H_c}(L^{-1})'$
- Previous spectral bounds extend to preconditioned matrix for $\mathcal{P}_{H_L}$ instead of $\mathcal{P}_{H_c}$
- Equivalent performance to SDP-based preconditioners



Closed-form preconditioner - results

Preconditioner computation time and the iterations required for cold-start convergence of the Fast Gradient Method

System	None	SDP		Proposed	
	Iter.	Iter.	Design Time (ms)	Iter.	Design Time (ms)
Schur-stable	42	16	197.4	16	0.213
Ill-conditioned Schur-stable	294	32	142.9	31	0.218
Inverted pendulum (non-prestabilized)	143	129	18.69	(Not computable)	
Inverted pendulum (LQR prestabilized)	18	17	17.45	18	0.218
Distillation column (non-prestabilized)	97	43	151746	43	2.543
Distillation column (LQR prestabilized)	22	3	81929	3	2.545

Conclusions

Key Contributions

- Closed-form & horizon independent preconditioner
- Equivalent performance to SDP-based preconditioners

Future Directions

- Examine preconditioning of the dual QP for MPC
- Possibly use the controller K as a preconditioner

For more information

I. McInerney, E. C. Kerrigan, and G. A. Constantinides, "Horizon-independent Preconditioner Design for Linear Predictive Control," IEEE Transactions on Automatic Control, (Accepted, in-press). arXiv: 2010.08572.