Recent Advances in the OSQP Solver:

Differentiable Optimization, Accelerated Linear Algebra, and More

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With Goran Banjac and Rajiv Sambharya

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Contributors

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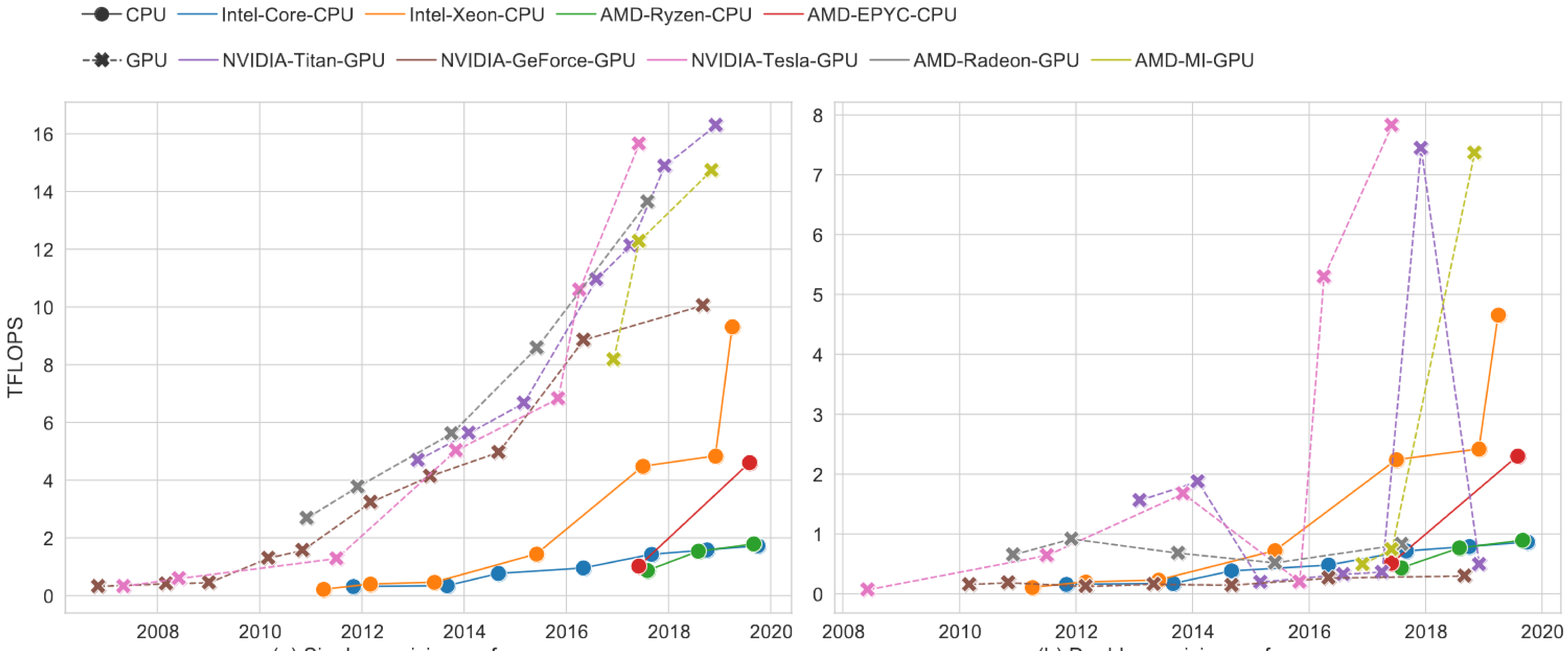
Tremendous progress in compute



(a) Single-precision performance.

(b) Double-precision performance.

[Y. Sun, N. B. Agostini, S. Dong, and D. Kaeli, “Summarizing CPU and GPU Design Trends with Product Data”, 2020, arXiv:1911.11313v2]

123.74
145
140
135
130
125
Pct. chg
Day
Chg.
1.64
1.98
Tues.
142.78

First-order methods

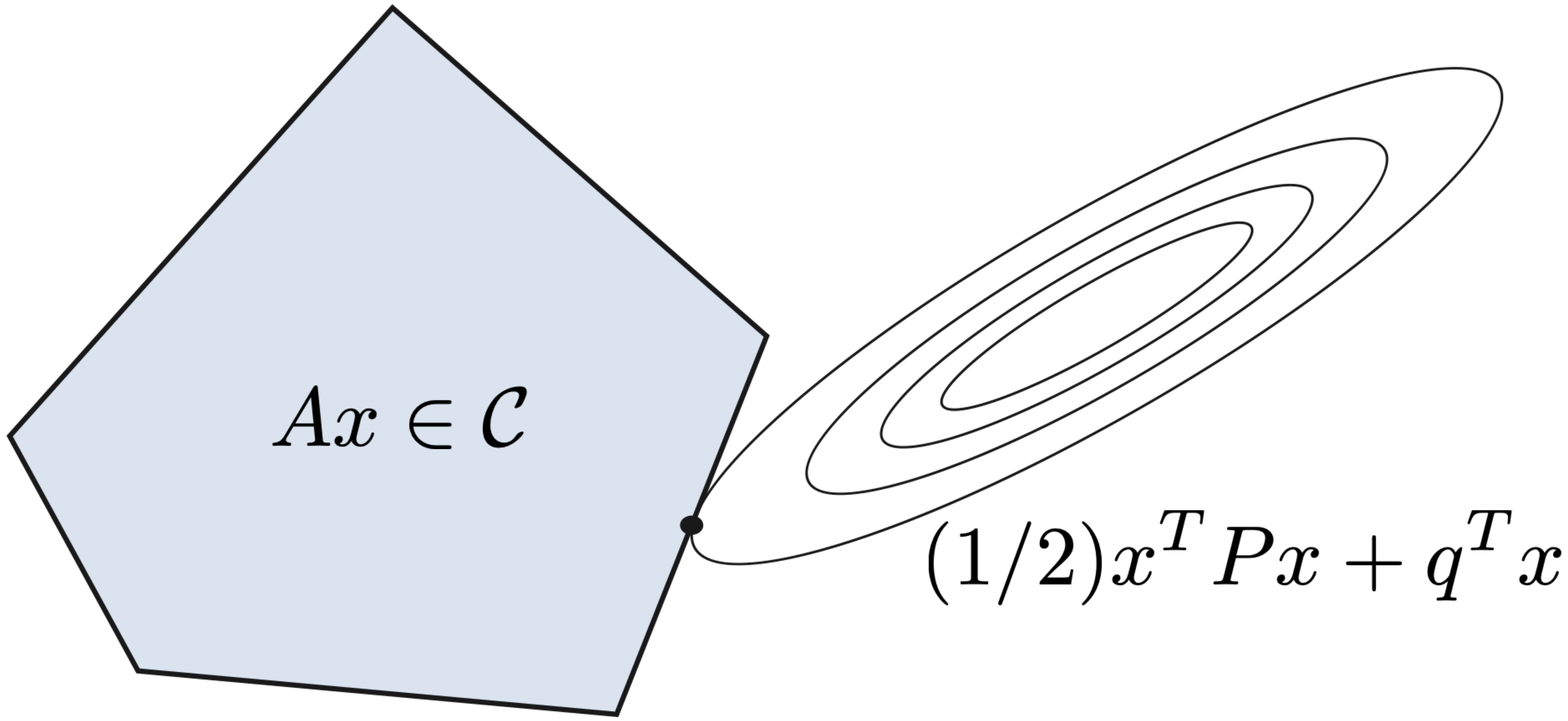
Wide popularity

Pros	Cons	OSQP
Warm-starting	Low quality solutions	High-quality solutions
Large-scale problems	Can’t detect infeasibility	Detects infeasibility
Embeddable	Problem data dependent	Robust

The problem

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x \\ \text{subject to} & Ax \in \mathcal{C} \end{array}$$

Quadratic program: $\mathcal{C} = [l, u]$



ADMM

Alternating Direction Method of Multipliers

Splitting

minimize $\quad f(x) + g(x) \quad \longrightarrow$

$$\begin{array}{ll} \text{minimize} & f(\tilde{x}) + g(x) \\ \text{subject to} & \tilde{x} = x \end{array}$$

Iterations

$$\tilde{x}^{k+1} \leftarrow \underset{\tilde{x}}{\text{argmin}} \left(f(\tilde{x}) + \rho/2 \left\| \tilde{x} - (x^k - y^k/\rho) \right\|^2 \right)$$

$$x^{k+1} \leftarrow \underset{x}{\text{argmin}} \left(g(x) + \rho/2 \left\| x - (\tilde{x}^{k+1} + y^k/\rho) \right\|^2 \right)$$

$$y^{k+1} \leftarrow y^k + \rho \left(\tilde{x}^{k+1} - x^{k+1} \right)$$

How do we split the QP?

$$\begin{array}{lll} \text{minimize} & (1/2)x^T P x + q^T x & \\ \text{subject to} & Ax = z & f \\ & z \in \mathcal{C} & g \end{array}$$

Splitting formulation

$$\begin{array}{ll} \text{minimize} & \underbrace{(1/2)\tilde{x}^T P \tilde{x} + q^T \tilde{x} + \mathcal{I}_{Ax=z}(\tilde{x}, \tilde{z})}_{f} + \underbrace{\mathcal{I}_{\mathcal{C}}(z)}_{g} \\ \text{subject to} & \tilde{x} = x \\ & \tilde{z} = z \end{array}$$

Diagonal
step sizes

σ

ρ

Complete algorithm

Problem

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x \\ \text{subject to} & l \leq Ax \leq u \end{array}$$

Algorithm

Linear system solve

$$(x^{k+1}, \nu^{k+1}) \leftarrow \text{solve} \begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho} I \end{bmatrix} \begin{bmatrix} x^{k+1} \\ \nu^{k+1} \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix}$$

Easy operations

$$\tilde{z}^{k+1} \leftarrow z^k + (\nu^{k+1} - y^k)/\rho$$

$$z^{k+1} \leftarrow \Pi\left(\tilde{z}^{k+1} + y^k/\rho\right)$$

$$y^{k+1} \leftarrow y^k + \rho\left(\tilde{z}^{k+1} - z^{k+1}\right)$$

Solving the linear system

Direct method (small to medium scale)

Quasi-definite matrix

$$\begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho} I \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix}$$

Well-defined LDL^T factorization

Factorization caching

$\longrightarrow$ **QDLDL**
Free quasi-definite linear system solver

[https://github.com/osqp/qdldl]

Solving the linear system

Indirect method (large scale)

Positive-definite matrix

$$\left(P + \sigma I + \rho A^T A\right) x = \sigma x^k - q + A^T(\rho z^k - y^k)$$

Conjugate gradient

Solve very large systems

GPU & FPGA implementation

Complete algorithm

Problem

minimize $\quad (1/2)x^T P x + q^T x$

subject to $\quad l \leq Ax \leq u$

Algorithm

Linear system solve

$$x^{k+1} \leftarrow \text{Solve} \quad (P + \sigma + \rho A^T A)x = \sigma x^k - q + A^T(\rho z^k - y^k)$$

Easy operations

$$z^{k+1} \leftarrow \Pi(Ax^{k+1} + \rho^{-1} y^k)$$

$$y^{k+1} \leftarrow y^k + \rho(Ax^{k+1} - z^{k+1})$$

always solvable!

OSQP

Operator Splitting solver for Quadratic Programs

Embeddable (can be division free!)

Supports warm-starting

Detects infeasibility

Solves large-scale problems

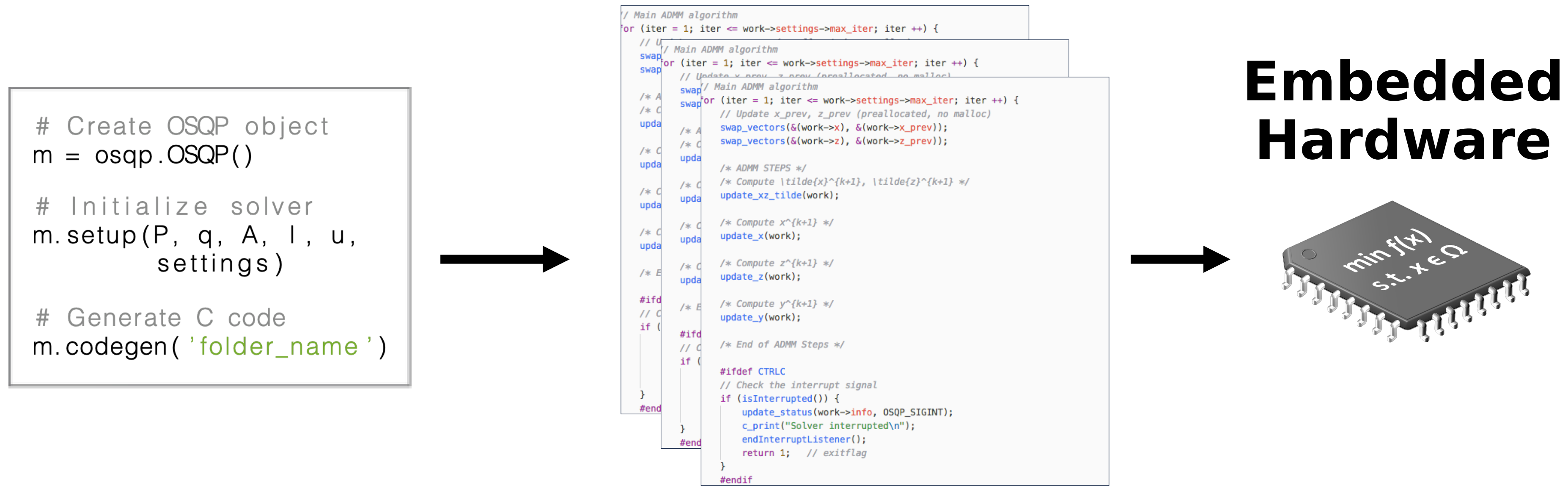
Users

More than 18 million downloads!

[pepy.tech/project/osqp]

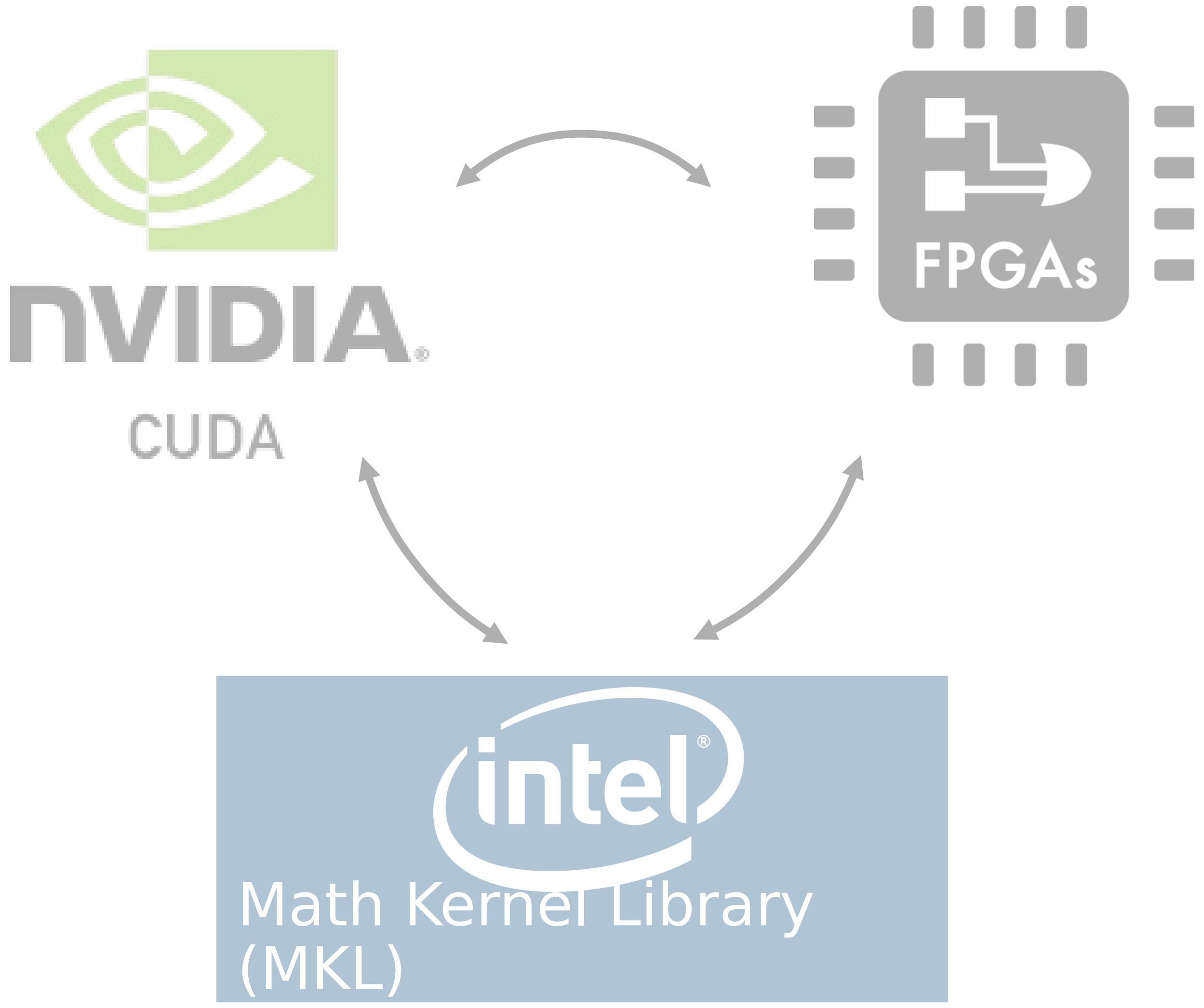
What’s new in OSQP 1.0

Improved embedded code generation



Code generation from C to C

Modular linear algebra



Differentiable layers

Code generation — Python API and result

Python API calls C code generation

```python
# Create an OSQP object
prob = osqp.OSQP()

# Setup workspace and change alpha parameter
prob.setup(P, q, A, l, u, alpha=1.0)

# Generate C code
prob.codegen(
    'folder',                   # Output folder for auto-generated code
    prefix='mysolver_',         # Prefix for filenames and C variables

    parameters='vectors',       # What do we wish to update in the generated code?
                                # 'vectors'/'matrices'
    use_float=False,            # Use single precision in generated code?
    printing_enable=False,      # Enable solver printing?
    profiling_enable=False,     # Enable solver profiling?
    interrupt_enable=False,     # Enable user interrupt (Ctrl-C)?
    include_codegen_src=True,   # Include headers/sources/Makefile in the output folder,
                                # creating a self-contained compilable folder?

    compile=False,              # Compile the python wrapper?
    python_ext_name='pyosqp',   # Name of the generated python extension
)
```

Naming

Solver options

Codegen wrapper

Code generation from C to C

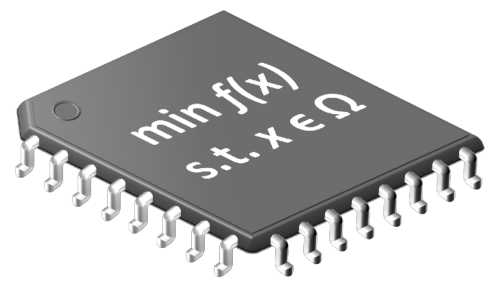
This is what gets called…

```
exitflag = osqp_setup(&solver, P, q, A, l, u, m, n, settings);

/* Test codegen */
OSQPCodegenDefines *defs = (OSQPCodegenDefines *)malloc(sizeof(OSQPCodegenDefines));

defs->float_type = 0;         /* Use doubles */
defs->printing_enable = 0;    /* Don't enable printing */
defs->profiling_enable = 0;   /* Don't enable profiling */
defs->interrupt_enable = 0;   /* Don't enable interrupts */

/* Generate code that allows only vector updates */
defs->embedded_mode = 1;
osqp_codegen(solver, vecDirPath, "vec_prefix_", defs);

/* Generate code that allows both vector and matrix updates */
defs->embedded_mode = 2;
osqp_codegen(solver, matDirPath, "mat_prefix", defs);
```

Desktop C solver

Embeddable code

- No dynamic memory allocation
- Division-free



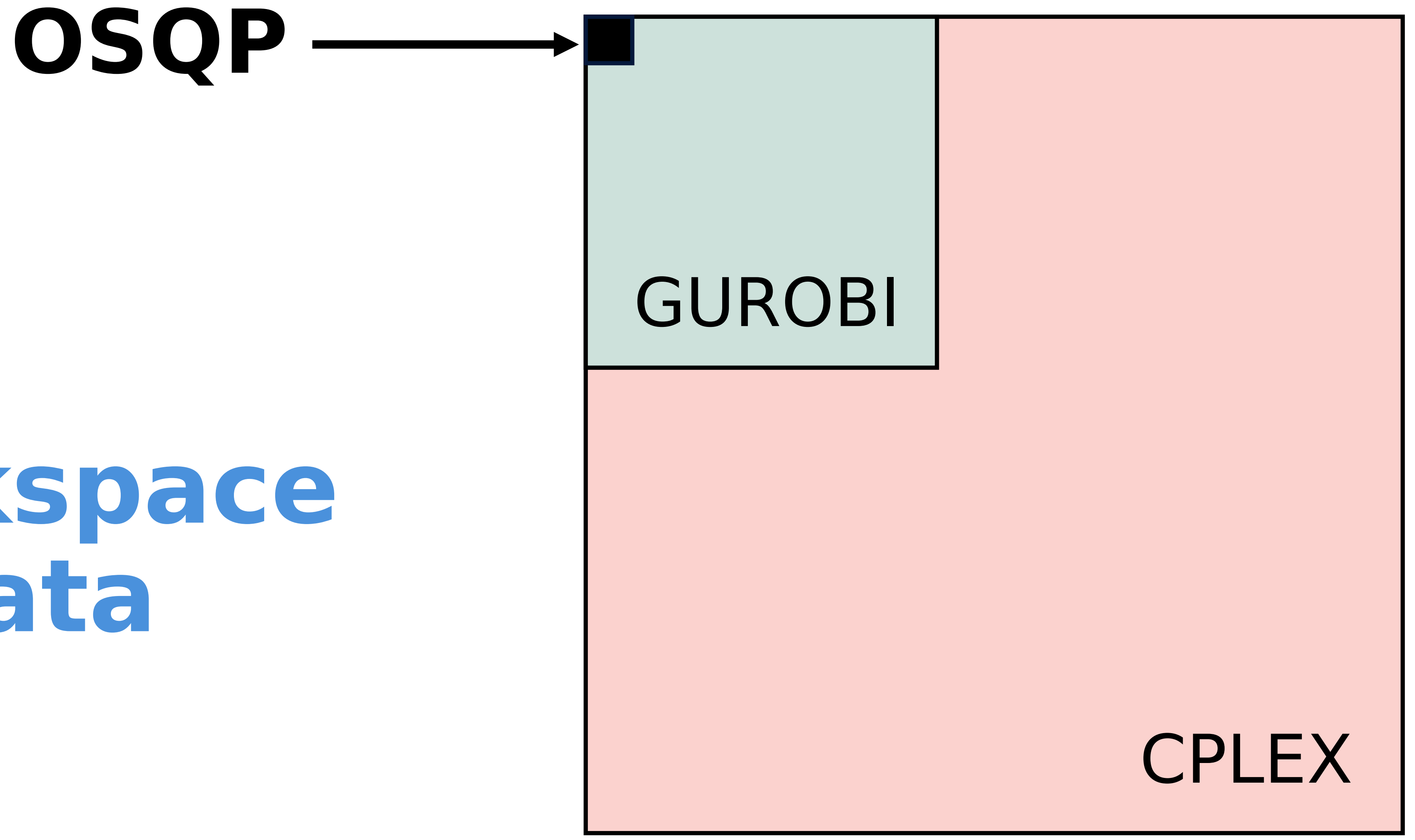
Code generation results

Self-contained and simplified directory structure

```
$ tree out
out
├── emosqp.c
├── inc
│   ├── private
│   │   ├── algebra_impl.h
⋮   ⋮   ⋮
│   │   └── version.h
│   └── public
│       ├── csc_type.h
│       ├── osqp_api_constants.h
│       ├── osqp_api_functions.h
│       ├── osqp_api_types.h
│       ├── osqp_api_utils.h
│       ├── osqp_export_define.h
│       └── osqp.h
├── Makefile
├── osqp_configure.h
├── mysolver_workspace.c
├── mysolver_workspace.h
└── src
    ├── algebra_libs.c
    ⋮
    ├── osqp_api.c
    ⋮
    └── vector.c
```

Workspace data

Compiled code size ~80kb (low footprint)

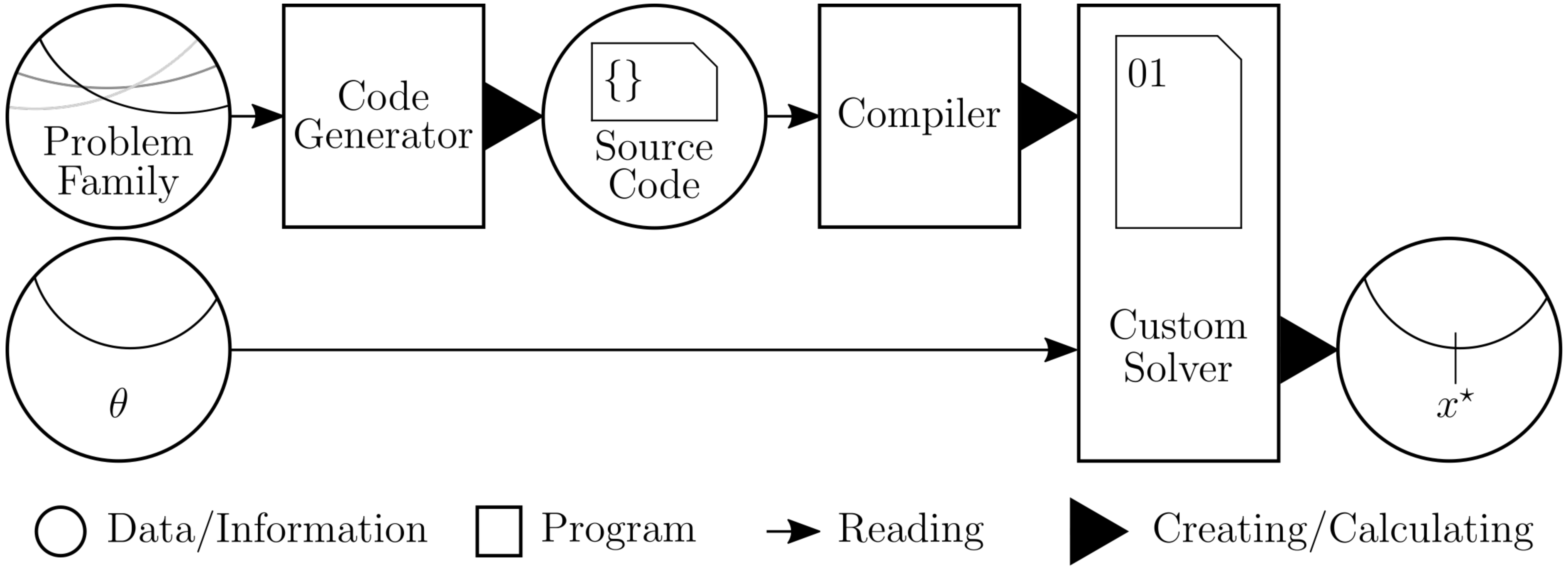


300x Reduction !

Code generation for parametric convex optimization

OSQP is integrated in CVXPYgen

https://pypi.org/project/cvxpygen/



Example

minimize $\|Gx - h\|^2$

subject to $x \geq 0$

$$\theta = (G, h)$$

```python
import cvxpy as cp
from cvxpygen import cpg

# model problem
x = cp.Variable(n, name='x')
G = cp.Parameter((m,n), name='G')
h = cp.Parameter(m, name='h')
p = cp.Problem(cp.Minimize(cp.sum_squares(G@x-h)),
               [x>=0])
# generate code
cpg.generate_code(p)
```

[Embedded Code Generation with CVXPY. Schaller, Banjac, Diamond, Agrawal, Stellato, and Boyd IEEE Control Systems Letters 2022]

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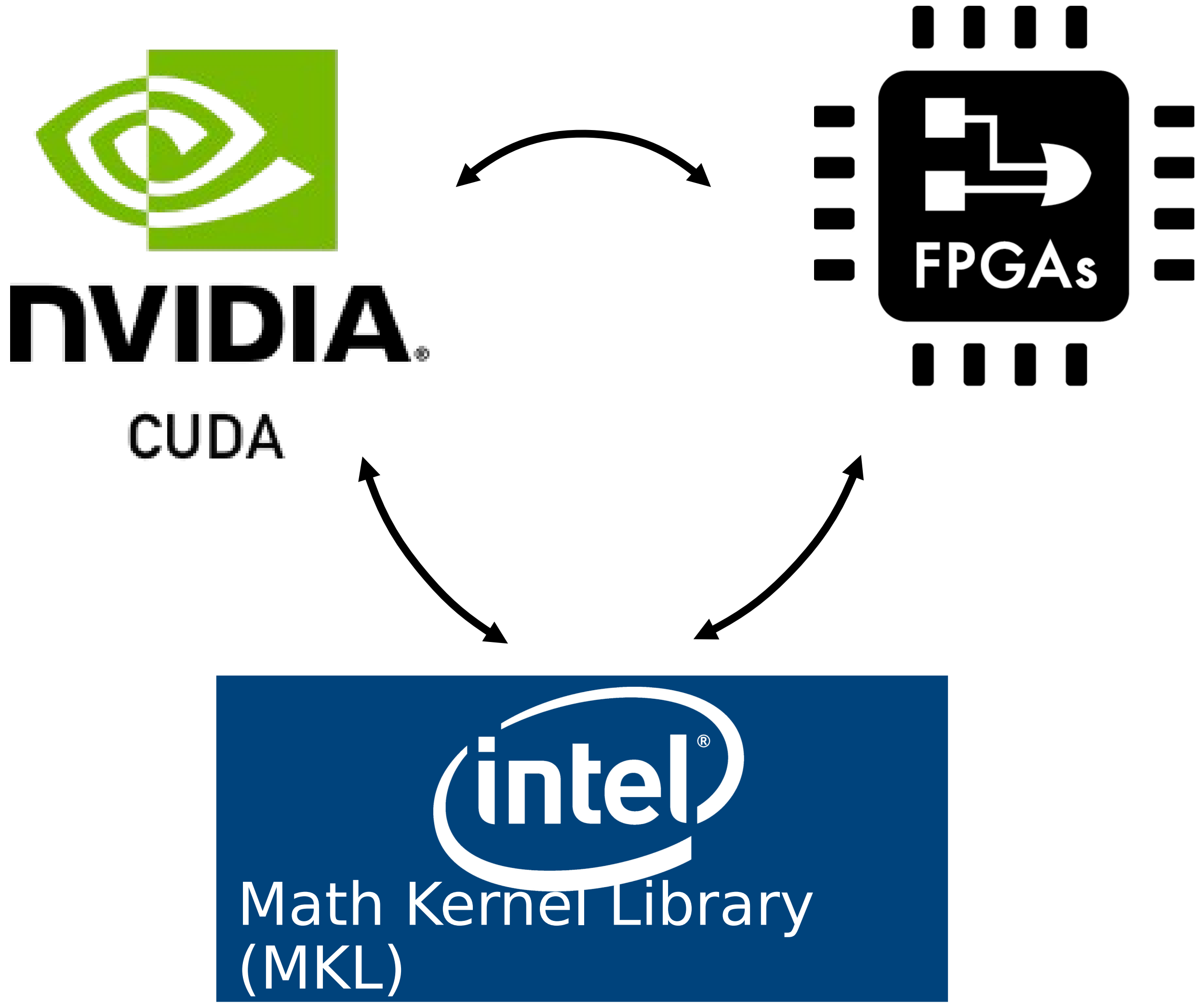
Code generation from C to C

Modular linear algebra

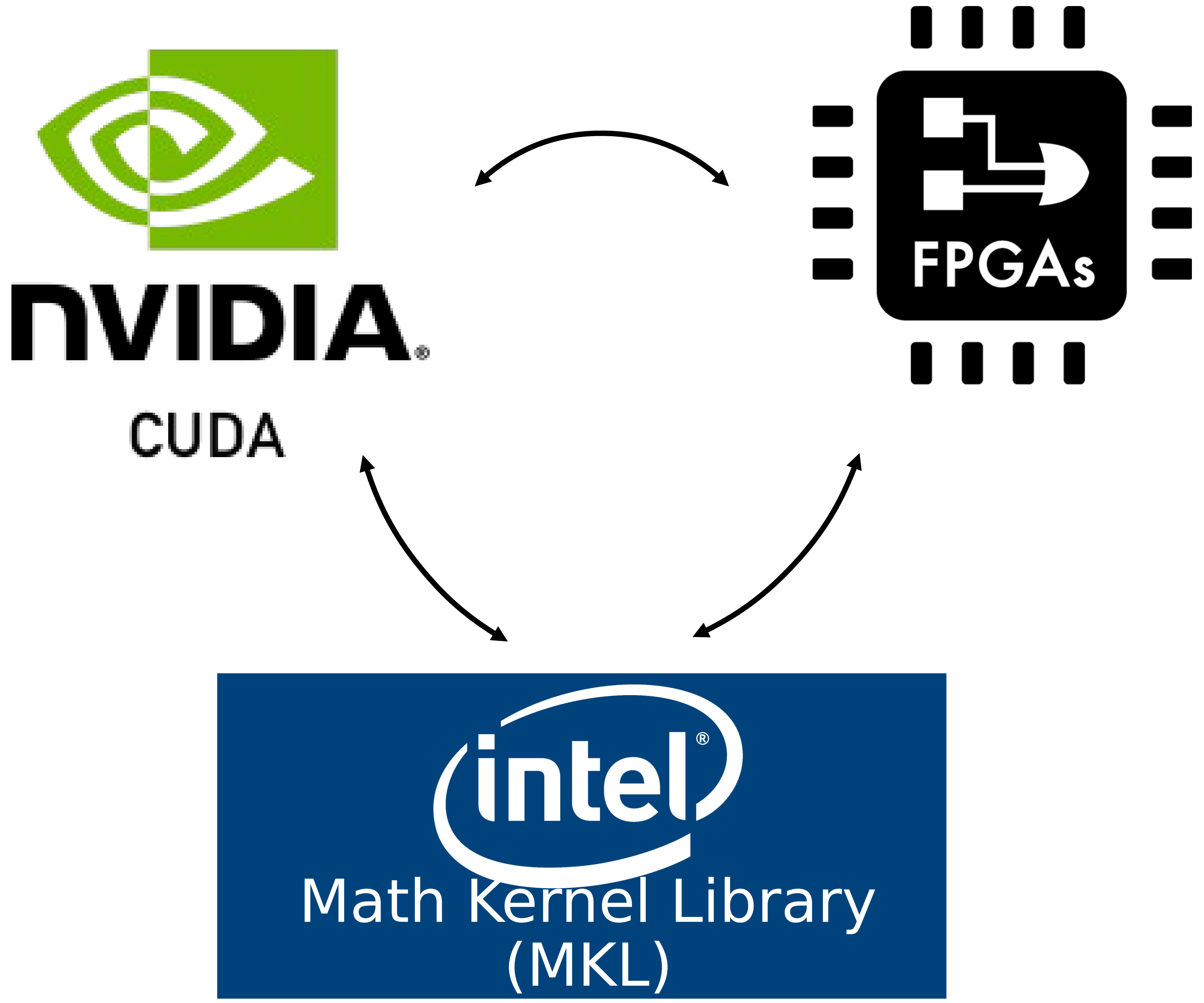


Differentiable layers

PyTorch

Modular Linear Algebra

Goal: easily switch between compute runtimes/systems



Modular Linear Algebra Backends

Available in 1.0:

- Standard CSC (hand-coded C)
- Nvidia CUDA[1]
- Intel MKL

Experimental:

- Sparse FPGA kernels[2]

Future:

- GraphBLAS
- Sycl/oneAPI
- ROCm
- …

[1] M. Schubiger, G. Banjac, and J. Lygeros, “GPU acceleration of ADMM for large-scale quadratic programming,” *Journal of Parallel and Distributed Computing*, vol. 144, pp. 55–67, 2020.
[2] M. Wang, I. McInerney, B. Stellato, S. Boyd, & H. Kwok-Hay So, “RSQP: Problem-specific Architectural Customization for Accelerated Convex Quadratic Optimization,“ *International Symposium on Computer Architecture (ISCA) 20*
Orlando, FL, USA, Jun. 2023. *(To appear).*

Modular Linear Algebra from Python

One-line import change

```python
# Import OSQP from a specific algebra backend module
from osqp.mkl import OSQP as OSQP_mkl
from osqp.cuda import OSQP as OSQP_cuda

prob_mkl = OSQP_mkl()
prob_cuda = OSQP_cuda()

# Setup workspace and change alpha parameter
prob_mkl.setup(P, q, A, l, u, alpha=1.0)

# Solve problem
res = prob_mkl.solve()
```

Setting in object constructor

```python
# Create an OSQP object with a specific algebra backend
if osqp.algebra_available('cuda'):
    # 'builtin' (default), 'mkl', or 'cuda'
    prob = osqp.OSQP(algebra='cuda')
else:
    prob = osqp.OSQP()

# Setup workspace and change alpha parameter
prob.setup(P, q, A, l, u, alpha=1.0)

# Solve problem
res = prob.solve()

...

# Solve with OSQP cuda on CVXPY
import cvxpy as cp

problm = cp.Problem(...)
problem.solve(solver=OSQP, algebra="cuda")
```

It works with CVXPY

Modular Linear Algebra from Julia

One-line import change

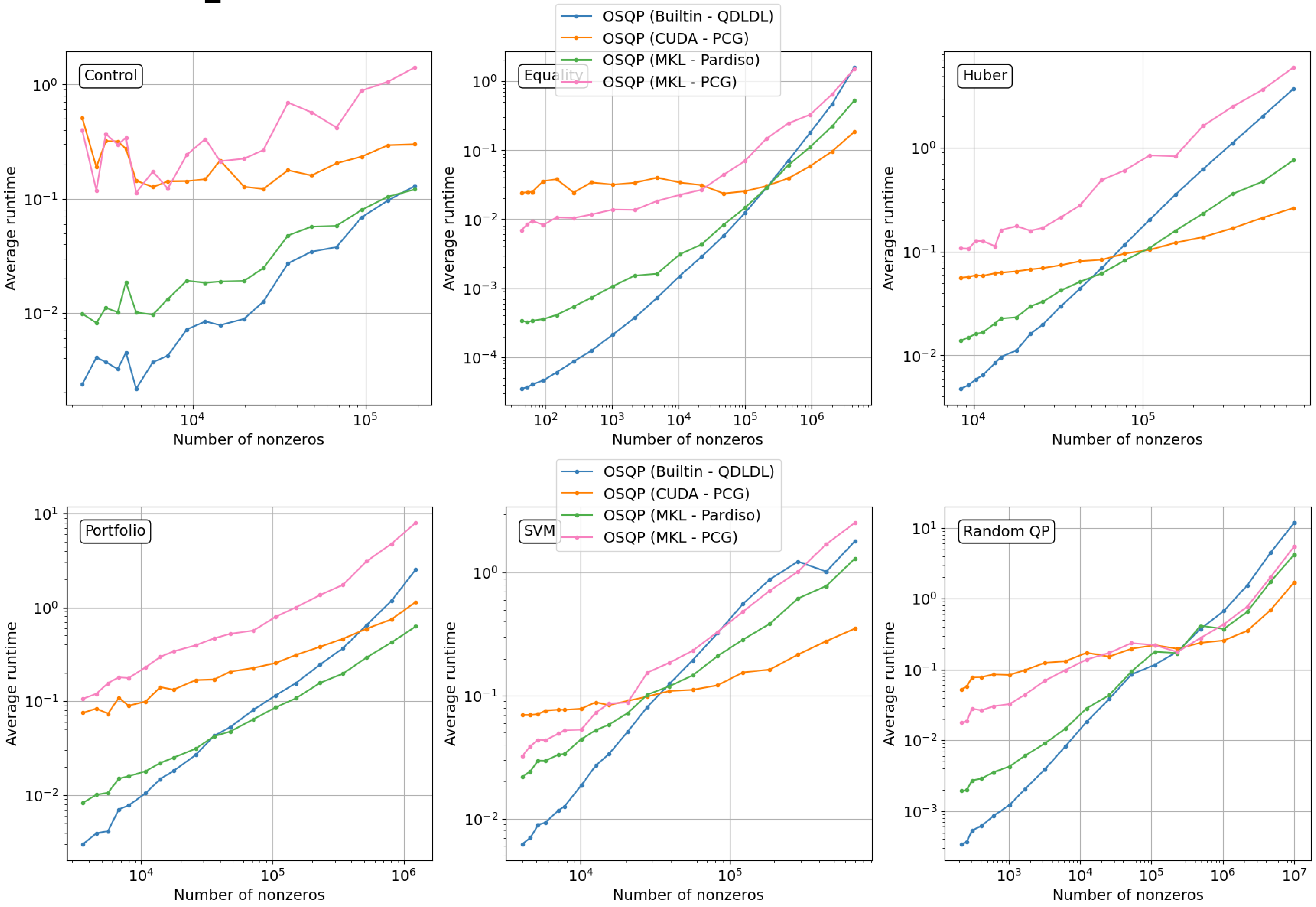
```julia
using JuMP
using OSQP
using OSQPMKL

model = Model( () -> OSQP.Optimizer(OSQPMKLAlgebra()) )

@variable(model, x >= 0)
@variable(model, 0 <= y <= 3)
@objective(model, Min, 12x + 20y)
@constraint(model, c1, 6x + 8y >= 100)
@constraint(model, c2, 7x + 12y >= 120)
print(model)
optimize!(model)
```

It works with JuMP

Comparisons of different algebras



$\mathbf{nnz} > 10^5$

CUDA is much faster

CPU: Intel Xeon W-2255, 3.70GHz

CUDA GPU: NVIDIA T1000

MKL: AVX512 support with 10 threads

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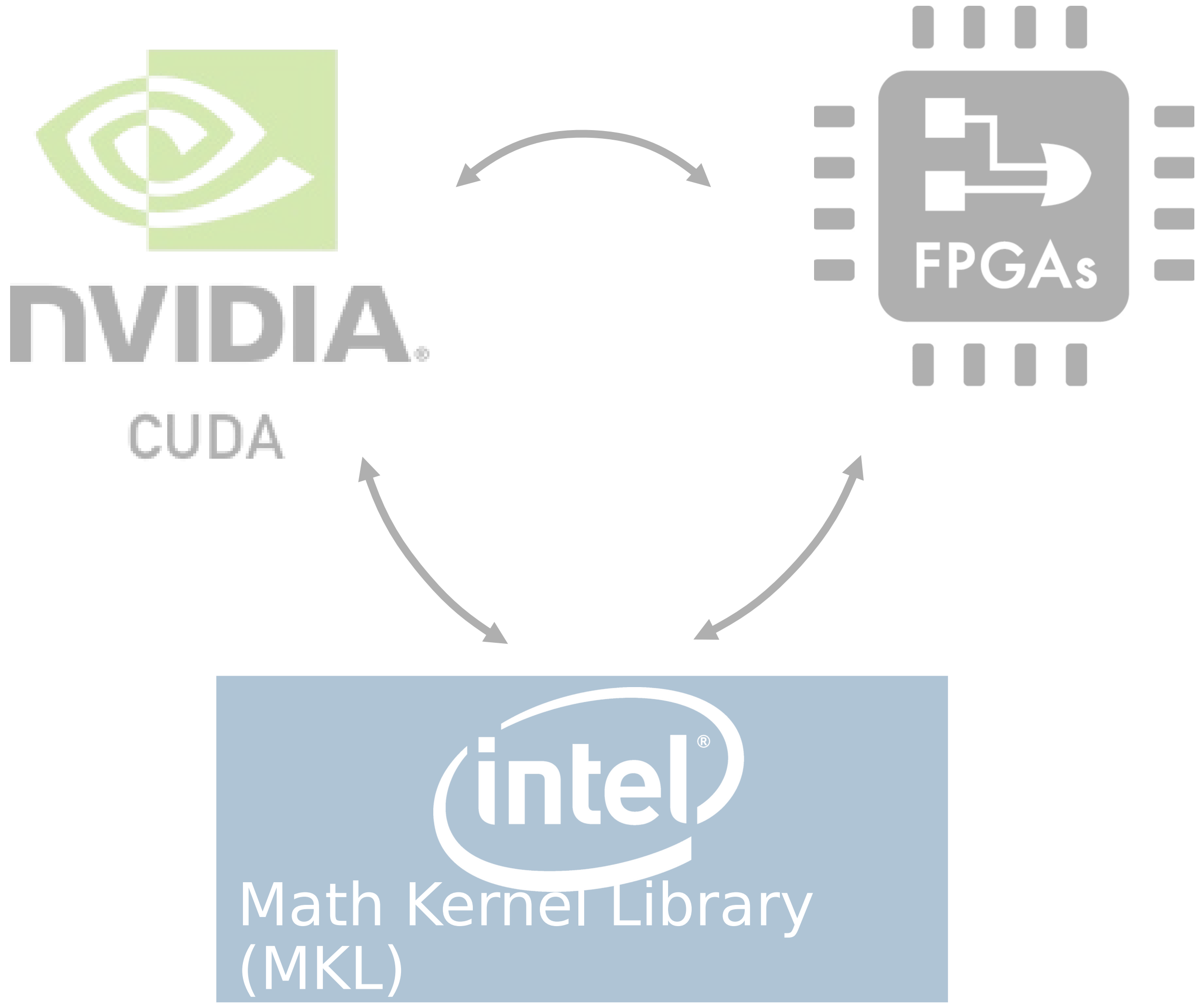

Differentiable layers

PyTorch

Derivatives computation in C

PyTorch

Implicit neural network architecture

Predictions $\xrightarrow{\theta = (P, q, A, l, u)}$

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P x + q^T x \\
\text{subject to} & l \leq Ax \leq u
\end{array}
$$

$\xrightarrow{x^\star(\theta)}$ Loss

Can model decision-making and constraints → **Many applications** control, robotics, optimal-transport, meta-learning…

For learning, we need to compute derivatives (backpropagate)

$$Dx^\star(\theta)$$

However, no QP solver supports derivatives internally (from C)!

Differentiating through QPs

$$\begin{array}{ll} \text{minimize} & (1/2)x^TPx + q^Tx \\ \text{subject to} & Ax \leq b \end{array}$$

- x variable
- $\theta = (P, q, A, b)$ parameter

Optimality conditions

stationarity

complementary slackness

$$\left.\begin{array}{r} \boxed{Px^\star + q + A^Ty^\star = 0} \\ \boxed{\mathbf{diag}(y^\star)(Ax^\star - b) = 0} \end{array}\right] \longrightarrow F(z^\star(\theta), \theta) = 0$$

$$Ax^\star - b \leq 0$$

$$y^\star \geq 0$$

primal/dual solution

$$z^\star = (x^\star(\theta), y^\star(\theta))$$

Goal

Compute $Dz^\star(\theta)$

Differentiating through convex optimization problems

$$F(z^\star(\theta), \theta) = 0$$

primal/dual solution

$$z^\star = (x^\star(\theta), y^\star(\theta))$$

Implicit function theorem

$$D_z F(z^\star, \theta) D z^\star(\theta) + D_\theta F(z^\star, \theta) = 0$$

$$\downarrow$$

$$D z^\star(\theta) = -\left(D_z F(z^\star, \theta)\right)^{-1} D_\theta F(z^\star, \theta)$$

($D_z F(z^\star, \theta)$ must be invertible) **linear system solution**

We plug $Dz^\star(\theta)$ in AD (automatic differentiation)

[Differentiating through a cone program. Agrawal, Barratt, Boyd, Busseti, Moursi. Journal of Applied and Numerical Optimization 2019]

[Differentiable Optimization-Based Modeling for Machine Learning. Amos. PhD Thesis 2019]

Derivatives computation directly in C

Import and define Pytorch layer

```python
from osqp.nn import OSQP  # Import Torch module

...

# Initialize NN layer with sparsity pattern
qp_layer = OSQP(P_idx, P_shape, A_idx, A_shape)

...

# Define architecture
x_star = qp_layer(P, q, A, l, u)

...

#  Loss based on x_star
l = loss(x_star)
```

Inside, it is calling this

```c
OSQPInt osqp_adjoint_derivative_compute(OSQPSolver* solver,
                                        OSQPFloat*  dx,
                                        OSQPFloat*  dy_l,
                                        OSQPFloat*  dy_u)
                                        { ... }

OSQPInt osqp_adjoint_derivative_get_mat(OSQPSolver*    solver,
                                        OSQPCscMatrix* dP,
                                        OSQPCscMatrix* dA);
                                        { ... }

OSQPInt osqp_adjoint_derivative_get_vec(OSQPSolver* solver,
                                        OSQPFloat*  dq,
                                        OSQPFloat*  dl,
                                        OSQPFloat*  du);
                                        { ... }
```

Still work in progress!

(To be integrated with CVXPYLayers and Flux.jl)

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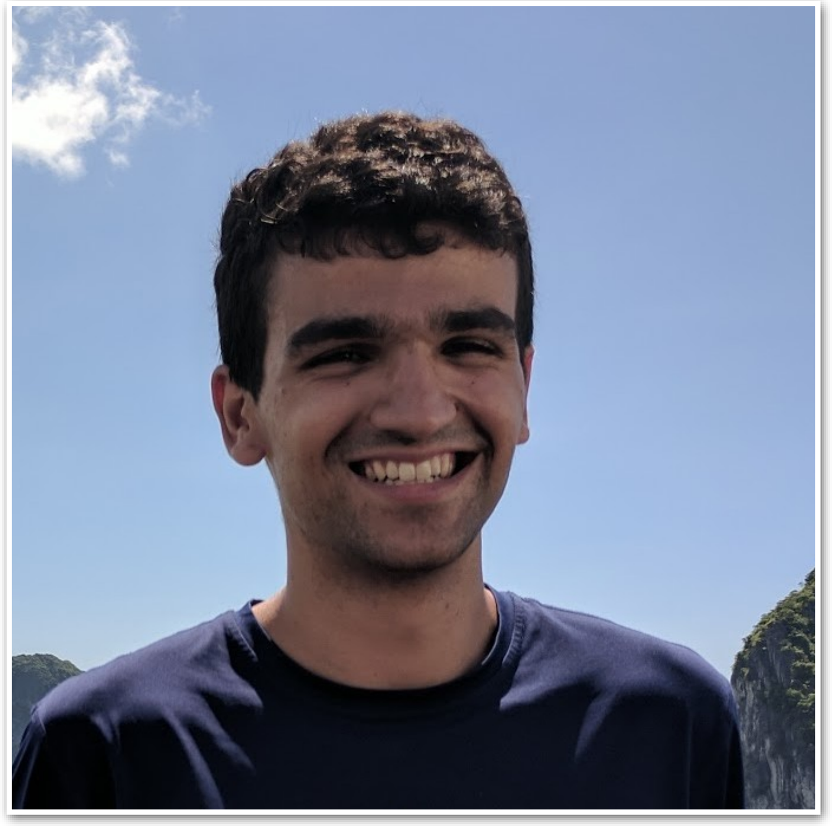

OSQP 1.0 – Beta Released!

20 minutes ago

imciner2

v1.0.0.beta0

3b3d162

Compare

v1.0.0.beta0 Pre-release

First beta release of OSQP v1.0

New features:

- Introduced new linear algebra backend system allowing compute framework to be changed at compile time.
- Merged cuOSQP project into main OSQP project (inside algebra/cuda directory).
- Introduced an Intel MKL-based algebra backend using the MKL sparse BLAS API, Vector Math Library. This backend contains both the Pardiso solver and an RCI conjugate gradient implementation.
- Added code generation capabilities to the C-level API (note, only problem export is in the C API, no file copying is done).
- Added initial adjoint derivative computation to the C-level API.

Main changes:

[github.com/osqp/{osqp,osqp-python,OSQP.jl}](https://github.com/osqp/)